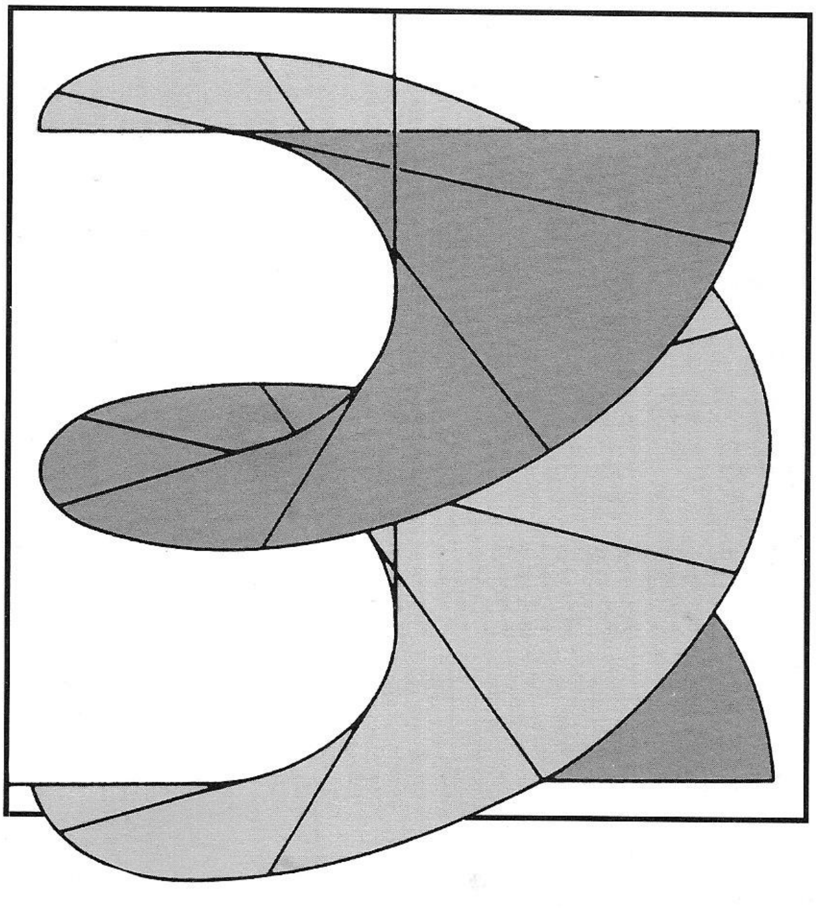
M434 Part II
Mathematics: A Fourth Level Course

M434
DIFFERENTIAL GEOMETRY



PART II
FRAME FIELDS AND CURVES

The Open University

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Mathematics: A Fourth Level Course

M434 Differential Geometry

Part II Frame Fields and Curves

Prepared for the Course Team
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Set book

Barrett O'Neill, *Elementary Differential Geometry*, hardback edition (Academic Press, 1966).

It is essential to have this book; the course is based on it and will not make sense without it.

The set book is referred to as *O'Neill*.

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Introduction

This part of the course begins the study of curves, the first application of the ideas introduced in Part I.

The second application, to surfaces, begins in Part IV.

There are a few geometric preliminaries which are needed and these are dealt with in Sections 1 and 2. *O'Neill* assumes that the idea of dot and cross product of vectors is understood already; what is done now is formally to transfer these operations to tangent vectors. We shall need these operations for investigating curves because we shall be using the 'moving frames' method. We shall need to be able to do calculations with tangent vectors at each point on the route of a curve in $\mathbf{E}^3$. These first two sections also give a reminder of the link between distances, as usually measured in $\mathbf{E}^3$, and dot products.

In Sections 3 and 4 the study of curves proper begins. First of all a carefully chosen set of basis vectors is placed at each point on the curve. Next, the rates of change of these basis vectors are found and expressed in terms of the basis. The formulas expressing the rates of change of the basis as linear combinations of the basis vectors are of central importance in the study of curves. The formulas are known as the Frenet (sometimes Serret–Frenet) formulas.

Where it seems harmless we shall refer to 'a point on the curve' where we should really refer to 'a point on the route of the curve'.

We shall eventually prove two theorems: the first says, essentially, that congruent curves have the same coefficients in their respective Frenet formulas. The second is the converse: that curves with the same coefficients in their Frenet formulas must be congruent.

We shall formally define congruence later but the formal definition will simply reflect intuition.

These theorems provide a strong link between the calculus of curves and their geometry. The Frenet formulas arise from the calculus by considering the rate of change of a basis. However, they completely determine the geometry of the curve. Perhaps the following, very non-rigorous suggestion may indicate that this is reasonable. Imagine driving along a road at constant speed. The forces that you experience pushing you from side to side (which are related to acceleration—a calculus notion) depend on how sharply curved the road is, compared with the speed with which you drive. Thus links between geometry and calculus are part of everyday experience.

Having shown, in the context of curves, that rates of change of basis vectors can provide geometric information, Sections 5–8 generalize the idea. Instead of just considering bases attached to curves, we consider a set of three vector fields providing an orthonormal basis at each point of $\mathbf{E}^3$. Such a set of vector fields is a frame field. We go on to show how the rate of change of a frame field may be expressed in terms of the frame field itself by using the notion of a 1-form introduced in Part I. Frenet-like formulas appear here, too. These ideas will not be fully exploited until we come to study surfaces.

Section 9 provides the usual end-of-chapter summary.

Study advice

The following represents a possible plan for study weeks.

Week 1 *O'Neill*, Chapter II, Sections 1–2 and TMA 01.

Week 2 *O'Neill*, Chapter II, Sections 3–5.

Week 3 *O'Neill*, Chapter II, Sections 6–8.

This plan leaves two study weeks for the sections of Part III required for TMA02 and for doing the TMA.

Sections 5–8 contain more general ideas than do Sections 1–4, and you may find them rather harder going.

1 Dot product

Read O'Neill: Chapter II, Section 1, pages 42–48.

You have probably already used dot and cross products of vectors to discuss orthogonality. In this section we formally transfer these ideas from vectors to tangent vectors.

The section begins with a review of these products for vectors. The properties on page 43 are probably familiar, even if you have not consciously thought about them.

Norm You may have used the notation $|\mathbf{p}|$ instead of $||\mathbf{p}||$ for norms. The double vertical bars for norms are used in this course, reserving single bars for the modulus function.

The norm of $\mathbf{p}$ is the same as the distance from the origin of the point represented by $\mathbf{p}$.

To calculate the dot or cross product of two tangent vectors at a particular point, we simply do the usual calculations with their vector parts, as Definition 1.3 shows for dot products.

There are alternatives to the approach in *O'Neill*. One such approach assumes that angles can be defined independently and *defines*

$$\mathbf{p} \cdot \mathbf{q} = ||\mathbf{p}||\,||\mathbf{q}|| \cos\theta.$$

The approach in *O'Neill* is the more usual one and $\mathbf{p} \cdot \mathbf{q}$ is used to define the notion of angle.

From this definition, it is fairly easy to show that

$$\mathbf{p} \cdot \mathbf{q} = p_1q_1 + p_2q_2 + p_3q_3.$$

Definition 1.4 This is very important. We shall be studying curves by placing a frame at each point on the curve.

Anticipating Section 5 a little, the vector fields that place a frame at each point form a frame field.

Orthonormal expansion The second part of Theorem 1.5 shows why we concentrate on frames, that is an orthonormal basis at each point. It is usually an effort (involving the solution of simultaneous equations) to find the coefficients in the expression of a vector as a linear combination of given basis vectors. For an orthonormal basis, the expression

$$\mathbf{v} = \sum_{i=1}^{3} (\mathbf{v} \cdot e_i) e_i$$

shows that the coefficients are easy to find. This process of orthonormal expansion will be used repeatedly both for theory and calculations.

The language used for frames and orthogonal matrices needs to be noted carefully. If a *basis* is described as orthogonal then all you know is that the vectors are mutually perpendicular. If the basis consists of mutually perpendicular *unit* vectors, then it is described as *orthonormal.*

However, the term orthogonal, when applied to a *matrix*, means that the rows are *orthonormal.*

As we shall see, the columns are also orthonormal.

Transpose The notation that is used in the course for transpose may well be different from the one that you have used before. If ${}^tA$ is read as 'the transpose of A' you may find it a little easier to use and remember.

You may have used $\overline{A}$ or A^* for the transpose of A.

The result

$${}^tA\,A = I$$

follows from direct matrix multiplication using

$$e_i \cdot e_j = \delta_{ij};$$

you do not need to use a 'standard theorem of linear algebra'. In fact, ${}^tAA = I$ is an alternative *definition* of orthogonality for matrices.

We know, from the above equation, that the inverse of an orthogonal matrix A is also the transpose of A:

$$A^{-1} = {}^tA.$$

Since

$$A^{-1}A = I = AA^{-1},$$

it follows that

$$A\,{}^tA = I.$$

We can rewrite this as

$${}^t({}^tA)({}^tA) = I,$$

which shows that the transpose of A is orthogonal and so the columns of A are orthonormal.

Linearity of cross product Explicitly this means that

$$\mathbf{v} \times (\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2) = \alpha_1 \mathbf{v} \times \mathbf{w}_1 + \alpha_2 \mathbf{v} \times \mathbf{w}_2$$

and

$$(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2) \times \mathbf{w} = \alpha_1 \mathbf{v}_1 \times \mathbf{w} + \alpha_2 \mathbf{v}_2 \times \mathbf{w}.$$

This, like the dot product, is an example of bilinearity.

The cross product gives, as Lemma 1.8 shows, a method of constructing a vector orthogonal to two given vectors. We shall use this very often to obtain the third member of a frame after we have found two members. More generally, having found two independent vectors, the cross product will enable us to construct the third member of a basis (though not necessarily a frame).

The length formula for the cross product gives the sine form:

$$\begin{aligned}
||\mathbf{v} \times \mathbf{w}||^2 &= (\mathbf{v} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{w}) - (\mathbf{v} \cdot \mathbf{w})^2 \\
&= ||\mathbf{v}||^2 ||\mathbf{w}||^2 - (||\mathbf{v}||\,||\mathbf{w}|| \cos\theta)^2 \\
&= ||\mathbf{v}||^2 ||\mathbf{w}||^2 (1 - \cos^2\theta) \\
&= ||\mathbf{v}||^2 ||\mathbf{w}||^2 \sin^2\theta.
\end{aligned}$$

You may find it easier to visualize the 'right-hand rule', in the last paragraph of page 48, by thinking of a corkscrew in conjunction with the diagram at the top of the page. If the handle rotates from $\mathbf{v}$ to $\mathbf{w}$ then the corkscrew travels in the direction of $\mathbf{v} \times \mathbf{w}$.

It is *possible* that someone has produced left-handed corkscrews! We have only ever seen right-handed ones and it is this type that we suggest that you imagine.

Exercise 1.1 *O'Neill*, page 48, Exercise 1.

Exercise 1.2 *O'Neill*, page 49, Exercise 3.

Exercise 1.3 *O'Neill*, page 49, Exercise 4 (parts (a) and (d) only). You may use the other results to do part (d).

Exercise 1.4 *O'Neill*, page 49, Exercise 5.

Exercise 1.5 *O'Neill*, Page 49, Exercise 6.

Exercise 1.6 *O'Neill*, page 49, Exercise 7.

[*Solutions on page 25*]

2 Curves

Read *O'Neill: Chapter II, Section 2, pages 51–55.*

Erratum *O'Neill*, page 55, the displayed expression for α'' should read

$$\alpha'' = \left(\frac{d^2\alpha_1}{dt^2}, \frac{d^2\alpha_2}{dt^2}, \frac{d^2\alpha_3}{dt^2}\right).$$

This last expression is the same as $(\alpha_1'', \alpha_2'', \alpha_3'')$.

There are two main ideas introduced in this section. The first is that any curve may be reparametrized so as to have unit speed. The second is that it is useful to consider vector fields that are defined only for points on the (route of the) curve.

The proof that a unit-speed reparametrization of a curve can always be found is constructive: the proof provides a method for finding the reparametrization. Although we do ask that you are able to carry out this process in simple cases, we shall make most use of the *existence* of a unit-speed reparametrization rather than the actual reparametrization.

The idea of a vector field defined only for points on a curve is quite natural if you consider the idea of the velocity vector of a curve. The velocity vector $\alpha'(t)$ is a tangent vector at $\alpha(t)$. Thus

$$t \longmapsto \alpha'(t)_{\alpha(t)}$$

defines a tangent vector at each point on the curve, but nowhere else. This is the motivation behind the definition of 'vector field on a curve'.

Proof of Theorem 2.1 Since the derivative of $s(t)$ is positive, s is increasing on its domain and so is one–one. Hence s has an inverse.

Note that this proof provides a technique for finding a unit-speed reparametrization of a curve α. The process can be broken down into stages as follows.

(a) Find the velocity, α', of the curve.

(b) Find the speed, $v(t) = ||\alpha'(t)||$.

(c) Using a suitable 'base point', $\alpha(a)$, find the arc-length function

$$s(t) = \int_a^t v(u)\, du.$$

(d) Invert s to obtain $t = t(s)$.

(e) Find the unit-speed reparametrization $\beta(s) = \alpha(t(s))$.

Parallel vector fields Note that *O'Neill* slips in a definition of parallel vector field. It is defined only for vector fields on a curve but generalizes to the following.

> **Definition**
>
> A vector field on $\mathbf{E}^3$ is *parallel* if the vector part of $V(\mathbf{p})$ is constant.

Straight lines There is a consequence of the proof of Lemma 2.3: the definition of 'straight line' given by *O'Neill* is extremely restrictive. A curve is a straight line if, and only if, its velocity is non-zero and its acceleration is zero. Thus, many curves whose routes are straight lines, in the usual sense, are excluded from this definition. For example: if $\mathbf{p}$ and $\mathbf{v}$ are vectors in $\mathbf{E}^3$, with $\mathbf{v}$ non-zero, then

$$\alpha(t) = \mathbf{p} + t\mathbf{v}, \quad t \in \mathbf{R}$$

defines a straight line, whereas

$$\alpha(t) = \mathbf{p} + t^2\mathbf{v}, \quad t \in \mathbf{R}$$

does not because the acceleration is $2\mathbf{v}$ and is non-zero. However, both have the same route.

Exercise 2.1 This question concerns the curve defined by

$$\alpha(t) = (t\cos t, t\sin t, t), \quad t \in \mathbf{R}.$$

(a) Find the velocity, speed and acceleration of α.

(b) Show that the route of α passes through the origin and find the velocity, speed and acceleration at the origin.

Exercise 2.2 *O'Neill*, page 55, Exercise 3.

Exercise 2.3 *O'Neill*, page 55, Exercise 4.

Exercise 2.4 *O'Neill*, page 56, Exercise 8.

[*Solutions on page 26*]

3 The Frenet formulas

Read *O'Neill: Chapter II, Section 3, pages 56–63.*

This section is the first application of the 'method of moving frames'. We define a frame at each point on a curve and express the derivatives of the resulting vector fields in terms of the fields themselves.

Since we are dealing here with unit-speed curves, the velocity vector (field) provides a ready made unit vector field on the curve. The major step in getting useful information about curves is in defining the other two vector fields appropriately.

The work in this section is in three parts.

(a) The definitions of the vector fields and functions associated with a curve (the **Frenet apparatus** of the curve).

(b) The relationships between the derivatives of the vector fields and the fields themselves (the **Frenet formulas**).

(c) Examples of the use of the Frenet formulas.

The Frenet apparatus and formulas The treatment in *O'Neill* has the definitions, results and proofs interwoven. There follows a summary of the essential points with the definitions, results and proofs separated. You may find this summary rather easier to use later on, for revision.

This interweaving is reasonable from the teaching point of view but may make revision harder, hence our summary.

Definitions For a unit-speed curve

$$\beta : I \longrightarrow \mathbf{R}, \quad I \text{ an interval in } \mathbf{R},$$

with non-vanishing acceleration, we define the Frenet apparatus of β to be T, κ, N, B and τ, where

$T = \beta'$ is the unit tangent vector field,

$\kappa = ||T'||$ is the curvature,

$N = \dfrac{T'}{\kappa}$ is the unit normal vector field,

$B = T \times N$ is the unit binormal vector field,

$\tau = -B' \cdot N$ is the torsion.

Note: The definition of τ differs slightly from that in *O'Neill*; this is because the definition in *O'Neill* is inextricably bound up with the proofs of the formulas. Separating the definitions out requires the above. You will find that *O'Neill* makes reference to this definition of τ a little later on.

Frenet formulas With the assumptions and definitions as above

$$T' = \kappa N$$
$$N' = -\kappa T + \tau B$$
$$B' = -\tau N.$$

Note: You may care to remember these formulas in the following 'matrix' form.

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

This form has the advantage of clearly showing the patterns in the coefficients; it also has strong similarities to a generalization of the Frenet formulas that will be discussed in Section 7.

Proofs If you re-read the proofs in *O'Neill* carefully, you will see that they hinge on repeated use of two ideas: orthonormal expansion and the Leibniz property of differentiation. Specifically, we shall want to express a vector field, Y say, on β as

$$Y = (Y \cdot T)T + (Y \cdot N)N + (Y \cdot B)B.$$

This uses orthonormal expansion.

We shall also want to use

$$(Y \cdot Z)' = Y' \cdot Z + Y \cdot Z'$$

This is the Leibniz property.

for vector fields Y and Z on β.

Before we can use orthonormal expansion, we must be sure that T, N and B are orthonormal at each point. To do so we use the definitions of the Frenet apparatus. This is essentially the content of Lemma 3.1. We gather the scattered elements of the proof together here.

Proof of Lemma 3.1 Since β is unit speed, we have

$$||T|| = ||\beta'|| = 1.$$

Consider the consequences of the definitions of N and κ:

$$\begin{aligned} ||N|| &= \frac{||T'||}{\kappa} && \text{(definition of } N) \\ &= \frac{||T'||}{||T'||} && \text{(definition of } \kappa) \\ &= 1. \end{aligned}$$

Because $T \cdot T = 1$, we have

$$\begin{aligned} & T' \cdot T + T \cdot T' = 0 \quad \text{(Leibniz)} \\ \Rightarrow\ & 2T \cdot T' = 0 \\ \Rightarrow\ & T \cdot T' = 0. \end{aligned}$$

But now it follows that

$$T \cdot N = T \cdot \frac{T'}{\kappa} = 0.$$

Thus T and N are unit length and orthogonal at each point of β.

Finally,

$$B = T \times N$$

is, by definition of cross product, orthogonal to T and N. Also, since T and N are orthogonal,

$$||B|| = ||T||\,||N|| \sin\frac{\pi}{2} = 1.$$

This completes the proof that T, N and B form an orthonormal basis at each point of β.

Note how we needed to know that T and N were *orthogonal* unit vectors before we could deduce that

$$B = T \times N$$

was a unit vector.

With Lemma 3.1 and the definitions of the Frenet apparatus, we can actually prove all three of the Frenet formulas using orthonormal expansion. If we write out the orthonormal expansions of T', N' and B', we obtain

$$\begin{aligned}
T' &= (T' \cdot T)T + (T' \cdot N)N + (T' \cdot B)B \\
N' &= (N' \cdot T)T + (N' \cdot N)N + (N' \cdot B)B \\
B' &= (B' \cdot T)T + (B' \cdot N)N + (B' \cdot B)B.
\end{aligned}$$

The definition

$$N = \frac{T'}{\kappa}$$

shows that the first expansion reduces to

$$T' = \kappa N,$$

and so we can deduce that

$$T' \cdot T = 0$$

(which we have shown independently) and that

$$T' \cdot B = 0.$$

To prove the other two formulas, we must calculate the coefficients; it is here that we again use the Leibniz property.

We have already differentiated $T \cdot T = 1$ to show that

$$T' \cdot T = 0.$$

A similar argument shows that

$$N' \cdot N = 0 \quad \text{and} \quad B' \cdot B = 0.$$

Next we tackle $N' \cdot T$. We have

$$\begin{aligned}
& N \cdot T = 0 \\
\Rightarrow\ & N' \cdot T + N \cdot T' = 0 \\
\Rightarrow\ & N' \cdot T = -N \cdot T' \\
\Rightarrow\ & N' \cdot T = -N \cdot \kappa N = -\kappa N \cdot N = -\kappa.
\end{aligned}$$

A similar argument, starting from $N \cdot B = 0$, shows that

$$N' \cdot B = -B' \cdot N = -(-\tau) = \tau \quad \text{(using the definition of } \tau\text{)}.$$

Finally, starting from $B \cdot T = 0$ gives

$$B' \cdot T = -B \cdot T' = -0 = 0.$$

Assembling all this information gives

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

If we denote the above matrix of coefficients by

$$F = (f_{ij}), \quad i, j = 1, 2, 3,$$

then the entries f_{ij} have the property

$$f_{ji} = -f_{ij}, \quad i, j = 1, 2, 3.$$

We can express this in terms of the transpose as

$${}^tF = -F.$$

This 'skew-symmetry' arises from one property of the geometry and one of the calculus. The geometric property is that T, N and B are a frame (orthonormal basis) at each point of β.

The Leibniz property is then applied to

$$Y \cdot Z = \text{constant}$$

to give

$$Y' \cdot Z = -Y \cdot Z'$$

with Y and Z replaced by all possible choices of T, N and B.

It may well occur to you that any similar situation involving derivatives of frames will give rise to similar skew-symmetry amongst the coefficients. You would be correct; we shall consider one other such case in Section 7.

Finding Frenet apparatus The definitions of the Frenet apparatus are constructive. That is, they can be used to find the apparatus of a given unit-speed curve with non-vanishing acceleration. *O'Neill* has a worked example in Example 3.3 (pages 58–59). In this example τ could as easily be calculated from

$$\tau = -B' \cdot N$$

as by comparison of B' and N.

Curves in $\mathbf{E}^2$ We have set you one exercise on curves in $\mathbf{E}^2$. The situation for curves in $\mathbf{E}^2$ is somewhat simpler than for curves in $\mathbf{E}^3$ and the definition of κ is a little different. We shall not consider such curves further, although we shall consider curves in $\mathbf{E}^3$ which happen to lie in a plane.

We know, from *O'Neill*, Corollary 3.5, that $\tau = 0$ for a plane curve.

Interpretation of Frenet apparatus The vector (field) T on a curve is, by definition, bound up with the geometric notion of 'tangent'. The other items in the Frenet apparatus also have geometric interpretations.

The normal, N, is in the direction of the rate of change of the tangent, by virtue of

$$T' = \kappa N.$$

Firstly, this shows that the tangent is 'turning' towards N. If we were to approximate the curve by a circle near some point, then N would point towards the centre of the circle, that is, towards the concave side of the curve.

Secondly, this formula also shows that κ measures the magnitude of the rate of turning of the tangent. The rate at which the tangent turns is, intuitively, a sensible way of measuring 'curvature'. This is why κ is defined to be the curvature function.

What is not so obvious, but can be shown in various ways, is that the 'best approximation' to a curve near $\alpha(t)$ by a *circle* is the circle which has the same tangent to the curve at the point, lies in the T–N plane and has radius $1/\kappa$.

Intuitively, if you want a plane approximation to the curve near some point, then the T–N plane is where you will have to look. Since B is, by definition, perpendicular to this plane, the rate of change of B will measure how the plane approximation varies from point to point. It is not surprising, therefore, that Corollary 3.5 shows that $\tau = 0$, which is equivalent to $B' = 0$, is the correct condition for detecting curves which always lie in a plane.

Exercise 3.1 *O'Neill*, page 63, Exercise 1.

Exercise 3.2 *O'Neill*, page 63, Exercise 2.

Exercise 3.3 *O'Neill*, page 64, Exercise 5.

Exercise 3.4 *O'Neill*, page 64, Exercise 6.

Exercise 3.5 *O'Neill*, page 65, Exercise 8. (See comment above about curves in the plane.)

[*Solutions on page 27*]

4 Arbitrary-speed curves

Read *O'Neill: Chapter II, Section 4, pages 66–74.*

This section extends the Frenet formulas to arbitrary speed curves. This is an important extension for practical computational reasons. Although a unit-speed reparametrization always *exists* (for the curves that we consider), actually finding it is often impractical.

The work in this section falls into three parts.

(a) The definition of the Frenet apparatus for (regular) curves in general and the consequent versions of the Frenet formulas.

(b) Obtaining practical computational formulas for the Frenet apparatus of such curves.

(c) Examples of the use of the formulas.

General Frenet apparatus The definitions say very much what one might expect: at any point the Frenet apparatus is defined to be the Frenet apparatus of the unit-speed reparametrization at the corresponding point. It is common to include the speed v when discussing the Frenet apparatus.

General Frenet formulas Equally, the appearance of the speed

$$s'(t) = v(t)$$

is predictable, given the central role that the chain rule plays when considering the effect of 'change of variable' on derivatives. The 'correction factor' that appears when changing variable from s to t is precisely $s'(t) = v(t)$. If we have a function F defined on the curve, then

$$\frac{dF}{dt} = \frac{dF}{ds}s'(t) = \frac{dF}{ds}v(t).$$

Once again, you may find the formulas more memorable in matrix form.

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa v & 0 \\ -\kappa v & 0 & \tau v \\ 0 & -\tau v & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

In theory, the general Frenet formulas would enable the Frenet apparatus of a curve such as

$$\alpha(t) = (t, t^2, t^3), \quad t \in \mathbf{R}$$

to be calculated. However, the calculations would still be very messy, as the following might indicate.

It would not be possible to find the unit-speed parametrization for this curve in explicit form, as you might like to check by experiment. The resulting integral for s cannot be expressed in terms of 'elementary functions'.

First, we could find the velocity,

$$\alpha'(t) = (1, 2t, 3t^2).$$

From this we get

$$v(t) = \sqrt{1 + 4t^2 + 9t^4}.$$

Hence, using $\alpha' = vT$,

$$T = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}}(1, 2t, 3t^2).$$

Differentiating this expression for T in order to find N from

$$T' = \kappa v N$$

is rather unpleasant!

The next step in *O'Neill* is to provide formulas for the Frenet apparatus in terms of α and its first three derivatives.

Computational results The formulas provided by Theorem 4.3 get round the problem indicated above.

The method of proof is typical of all uses of the Frenet formulas: differentiate and then use the Frenet formulas to replace derivatives of the frame vectors by combinations of the vectors themselves. Note that the differentiations usually involve use of the Leibniz property in one or other of its manifestations.

Special curves At this point *O'Neill* introduces two special classes of curves: spherical images and cylindrical helices.

Note that we adopt the 'classic' plural of helix: helices, in spite of using the non-classical plural 'formulas'!

The spherical image of a curve is defined by ignoring the point of application of T 'transferring T to the origin' as *O'Neill* puts it. Thus, the coordinate functions of T are used to define the spherical image curve.

You have already met the circular helix which is a special case of the cylindrical helix. Note that, as Fig. 2.19 shows, the 'cylinder' in 'cylindrical helix' should *not* be taken to imply 'circular cylinder'. The cross-section curve can be any plane curve.

Exercise 4.1 *O'Neill*, page 74, Exercise 1(a).

Exercise 4.2 *O'Neill*, page 74, Exercise 2.

Exercise 4.3 *O'Neill*, page 74, Exercise 3(a).

Exercise 4.4 *O'Neill*, page 74, Exercise 6. (*Hint*: Use the Frenet formulas and the fact that the speed is constant to express the derivatives of α in terms of the Frenet apparatus. Then adapt the proof of Lemma 4.2.)

Exercise 4.5 *O'Neill*, pages 74–5, Exercise 8.

Exercise 4.6 *O'Neill*, page 75, Exercise 10. (*Hint*: A circular helix is a cylindrical helix whose cross-section curve is a circle.)

Exercise 4.7 *O'Neill*, page 75, Exercise 11.

Note: Use the results obtained in *O'Neill* for σ', σ'' and κ_σ and just calculate τ_σ.

Exercise 4.8 *O'Neill*, page 76, Exercise 12(a).

[*Solutions on page 29*]

5 Covariant derivatives

Read *O'Neill: Chapter II, Section 5, pages 77–80.*

Errata

1 On page 80, Exercise 1, the second line:

for 'Compute $\nabla_{\mathbf{v}} \ldots$' read 'Compute $\nabla_{\mathbf{v}_{\mathbf{p}}} \ldots$'

2 On page 80, Exercise 5, the second sentence would probably read better as

'Thus ∇W is the ...'. ■

We now pause in our study of curves to begin the generalization of the method of moving frames. With a curve, the three vector fields T, N and B that provide the Frenet frame are functions of the single variable used to define the curve. The derivatives of the Frenet frame are, therefore, just the ordinary derivatives of elementary calculus. On a curve there is only one way to go: along the curve.

This generalization will take up Sections 5–8, we shall return to curves in Part III.

In general, there will be the possibility of moving in various directions from a given point so we need the concept of a directional derivative for vector fields. The idea of **covariant derivative** provides what we need. It is the direct analogue for vector fields of the directional derivative for functions from $\mathbf{E}^3$ to $\mathbf{R}$.

The covariant derivative $\nabla_{\mathbf{v}_{\mathbf{p}}} W$ measures the initial rate of change of the value of W, as you set off from $\mathbf{p}$ with velocity $\mathbf{v}$. Because values of W are tangent vectors, $\nabla_{\mathbf{v}_{\mathbf{p}}} W$ is a tangent vector based at $\mathbf{p}$.

Lemma 5.2 This is useful as it relates the new form of derivative to one that you have already met: the directional derivative of a function. Paraphrasing: to differentiate a vector field, differentiate the coordinate functions; exactly what you are used to doing with the Frenet vector fields.

Theorem 5.3 You should be expecting this sort of result to appear after each definition of a new form of derivative: it states that the new derivative has appropriate linearity and Leibniz properties. Parts (3) and (4) are not, perhaps, what you might write down immediately if asked to produce Leibniz properties for $\nabla_{\mathbf{v}_{\mathbf{p}}} W$. However, they are actually inevitable!

For example, consider part (3). To differentiate the product fY with respect to $\mathbf{v}_{\mathbf{p}}$, we have to differentiate each of f and Y with respect to $\mathbf{v}_{\mathbf{p}}$. Since f is a function, its derivative will be $\mathbf{v}_{\mathbf{p}}[f]$. The derivative of Y will be $\nabla_{\mathbf{v}_{\mathbf{p}}} W$. Finally, if we remember that the end result must be a tangent vector based at $\mathbf{p}$, we know that Y in the first term and f in the second must be evaluated at $\mathbf{p}$.

To understand that last point, consider a 'first guess':

$$\mathbf{v}_{\mathbf{p}}[f]\, Y + f \nabla_{\mathbf{v}_{\mathbf{p}}} Y.$$

Warning: this first guess is **incorrect!**

This is entirely reasonable *except* that it gives a result of the wrong sort. Actually, the first term is a vector field and the second is undefined (being a function multiplied by a tangent vector).

Examples It is instructive to consider three special cases of covariant derivatives, those with respect to $U_1(\mathbf{p})$ etc.

First we calculate

$$\nabla_{(1,0,0)_{\mathbf{p}}} W,$$

for an arbitrary vector field W with coordinate functions w_1, w_2 and w_3.

Applying Lemma 5.2 gives

$$\nabla_{(1,0,0)_{\mathbf{p}}} W = (1,0,0)_{\mathbf{p}}[w_1]U_1(\mathbf{p}) + (1,0,0)_{\mathbf{p}}[w_2]U_2(\mathbf{p}) + (1,0,0)_{\mathbf{p}}[w_3]U_3(\mathbf{p}).$$

However, we know from Chapter I that

$$(1,0,0)_{\mathbf{p}}[w_i] = \frac{\partial w_i}{\partial x}(\mathbf{p}).$$

Thus,

$$\nabla_{(1,0,0)_{\mathbf{p}}} W = \left(\frac{\partial w_1}{\partial x}(\mathbf{p}), \frac{\partial w_2}{\partial x}(\mathbf{p}), \frac{\partial w_3}{\partial x}(\mathbf{p})\right)_{\mathbf{p}}.$$

Paraphrasing: to differentiate with respect to $U_1(\mathbf{p})$, partially differentiate with respect to x (and evaluate at $\mathbf{p}$).

It seems entirely reasonable that directional differentiation (with unit speed) in the x-direction should reduce to $\partial/\partial x$.

Similar calculations show that

$$\nabla_{(0,1,0)_{\mathbf{p}}} W = \left(\frac{\partial w_1}{\partial y}(\mathbf{p}), \frac{\partial w_2}{\partial y}(\mathbf{p}), \frac{\partial w_3}{\partial y}(\mathbf{p})\right)_{\mathbf{p}},$$

$$\nabla_{(0,0,1)_{\mathbf{p}}} W = \left(\frac{\partial w_1}{\partial z}(\mathbf{p}), \frac{\partial w_2}{\partial z}(\mathbf{p}), \frac{\partial w_3}{\partial z}(\mathbf{p})\right)_{\mathbf{p}}.$$

You might like to note that these results are entirely consistent with the results

$$(1,0,0)_{\mathbf{p}}[f] = \frac{\partial f}{\partial x}(\mathbf{p})$$

etc., that we obtained earlier.

Extension to vector fields *O'Neill* follows the usual pattern next by extending the definition of covariant derivative to allow differentiation with respect to a vector field. The method of extension is exactly the same as for directional derivatives.

The definition is essentially that of a composite function:

$$\nabla_V W : \mathbf{p} \longmapsto V(\mathbf{p}) \longmapsto \nabla_{V(\mathbf{p})} W.$$

All definitions based on the pointwise principle are, effectively, composite function definitions.

At each point, differentiate W with respect to the tangent vector provided by V. This gives a function producing a tangent vector at each point, that is a vector field. Thus the covariant derivative of a vector field *with respect to a vector field* is also a vector field.

Because of this last remark, the results in Corollary 5.4 are exactly what a 'first guess' might well produce. (Compare the remarks about Theorem 5.3 made above.)

Examples The three special cases discussed above yield much less involved formulas when extended to differentiation with respect to the natural frame field.

For example, $\nabla_{U_1} W$ is, by definition,

$$\nabla_{U_1} W : \mathbf{p} \longmapsto U_1(\mathbf{p}) \longmapsto \nabla_{(1,0,0)_{\mathbf{p}}} W.$$

Using what we obtained above gives

$$\nabla_{U_1} W = \frac{\partial w_1}{\partial x} U_1 + \frac{\partial w_2}{\partial x} U_2 + \frac{\partial w_3}{\partial x} U_3.$$

In other words: to differentiate with respect to U_1, partially differentiate with respect to x.

Similarly,

$$\nabla_{U_2} W = \frac{\partial w_1}{\partial y} U_1 + \frac{\partial w_2}{\partial y} U_2 + \frac{\partial w_3}{\partial y} U_3$$

$$\nabla_{U_3} W = \frac{\partial w_1}{\partial z} U_1 + \frac{\partial w_2}{\partial z} U_2 + \frac{\partial w_3}{\partial z} U_3.$$

Note that these results correspond exactly to $U_1[f] = \partial f/\partial x$ etc.

Calculation techniques To calculate a given covariant derivative, repeated application of linearity will reduce the problem to derivatives with respect to the natural frame field. These derivatives can then be found by straightforward partial differentiation.

An extension The definition of the covariant derivative $\nabla_{\mathbf{v_p}} W$ given in this section is fine when the vector field W is defined on the whole of a region of $\mathbf{E}^3$ surrounding the point $\mathbf{p}$ because we can be sure that $W(\mathbf{p}+t\mathbf{v})$ is actually defined near $\mathbf{p}$. However, we shall want to apply these ideas on surfaces, where the vector fields being differentiated are defined only for points on the surface. The line $\mathbf{p}+t\mathbf{v}$ will probably leave the surface immediately you move away from $\mathbf{p}$, so we need an alternative definition of covariant derivative.

The basis of the alternative definition is contained in Exercise 6, page 81 of *O'Neill*. However, the idea is important enough for us to want to look at it now, rather than leaving it as an exercise.

Suppose that we have a curve

$$\alpha : t \longmapsto \alpha(t), \quad t \in I,$$

and a vector field W that is defined in a region that includes the route of α. Then we can form the composite

$$W(\alpha) : t \longmapsto W(\alpha(t)).$$

This composite has an ordinary derivative at t, namely $(W(\alpha))'(t)$, which is a tangent vector at $\alpha(t)$.

We can also covariantly differentiate W with respect to the tangent $\alpha'(t)$ at $\alpha(t)$. This will also be a tangent vector based at $\alpha(t)$. It is reasonable to ask how these two derivatives are related. The answer is that they are exactly the same:

$$\nabla_{\alpha'(t)} W = (W(\alpha(t)))'.$$

The proof involves applying *O'Neill*, Chapter I, Lemma 4.6 three times, once to each of the coordinate functions of W.

This lemma is the corresponding result to *O'Neill*, Chapter I, 4.6 for directional derivatives of functions.

It is less important to understand all the details of the proof than to appreciate the result and its consequences. Suppose that we have a curve α with the following properties:

$$\alpha(0) = \mathbf{p}, \quad \alpha'(0) = \mathbf{v}.$$

Now apply the above result at $t = 0$:

$$\begin{aligned}\nabla_{\mathbf{v_p}} W &= \nabla_{\alpha'(0)} W \\ &= (W(\alpha(t)))'(0).\end{aligned}$$

Let us look at what this says a little more carefully. If we define

$$\beta(t) = \mathbf{p} + t\mathbf{v},$$

then the definition of covariant derivative becomes

$$\nabla_{\mathbf{v_p}} W = (W(\beta(t)))'(0).$$

Now, it is fairly clear that

$$\beta(0) = \mathbf{p} \quad \text{and} \quad \beta'(0) = \mathbf{v}.$$

What we now know is that the straight line β can be replaced by *any* curve which goes through the point $\mathbf{p}$ with velocity $\mathbf{v}$, that is any curve α such that

$$\alpha(0) = \mathbf{p}, \quad \alpha'(0) = \mathbf{v}.$$

When we come to deal with surfaces, this result will become the *definition* of covariant derivative.

Exercise 5.1 *O'Neill*, page 80, Exercise 1.

Exercise 5.2 *O'Neill*, page 80, Exercise 2 (parts (a), (c) and (e) only).

Exercise 5.3 *O'Neill*, page 80, Exercise 4.

Exercise 5.4 *O'Neill*, page 80, Exercise 5. Errata

Exercise 5.5 Let the curve α and the vector field W be defined by

$$\alpha(t) = (1 + \sin t, 3 - \sin t, t^2 + 2t - 1),$$
$$W = x^2 U_1 + yU_2.$$

Show that

$$\alpha(0) = (1, 3, -1) \quad \text{and} \quad \alpha'(0) = (1, -1, 2).$$

Calculate

$$W(\alpha(t))$$

and hence find

$$\nabla_{(1,-1,2)_{(1,3,-1)}} W.$$

Compare your answer with the one you gave for the first exercise in this set.

[*Solutions on page 32*]

6 Frame fields

Read *O'Neill: Chapter II, Section 6, pages 81–84.*

In the last section we discussed the notion of derivative, the covariant derivative, that we require for generalizing the Frenet formulas. In this section we formally define a **frame field**.

The definition was anticipated when we referred to U_1, U_2 and U_3 as the natural *frame field*. The U_i provide an orthonormal basis at each point of $\mathbf{E}^3$.

The basis that *M203* refers to as the standard basis.

The preamble to Definition 6.1 and the definition itself formalize the idea of a set of vector fields providing an orthonormal basis at each point.

Examples We strongly recommend that you construct your own versions of Figures 2.21–2.23 of *O'Neill*; it is always difficult to follow someone else's sketches of three-dimensional arrangements.

Lemma 6.3 What *O'Neill* means by 'immediate consequence' is the following.

If $\mathbf{p} \in \mathbf{E}^3$, then $V(\mathbf{p})$ can be expressed in terms of the $E_i(\mathbf{p})$ by orthonormal expansion:

$$V(\mathbf{p}) = (V(\mathbf{p}) \cdot E_1(\mathbf{p}))E_1(\mathbf{p}) + (V(\mathbf{p}) \cdot E_2(\mathbf{p}))E_2(\mathbf{p}) + (V(\mathbf{p}) \cdot E_3(\mathbf{p}))E_3(\mathbf{p}).$$

By invoking the pointwise principle, we obtain the 'functional' form of the above:

$$V = (V \cdot E_1)E_1 + (V \cdot E_2)E_2 + (V \cdot E_3)E_3.$$

The expressions $V \cdot E_i$ define the coordinate functions f_i of V with respect to the frame field E_i.

Exercise 6.1 *O'Neill*, page 84, Exercise 1.

Note: This result is, in theory, quite useful. It gives a method of starting with two vector fields that are merely linearly independent and converting them into orthogonal unit vector fields. (The third member of a frame is then generated using the cross product.) In practice we shall use this technique only on one occasion, in connection with surfaces.

Exercise 6.2 *O'Neill*, page 84, Exercise 2, for the cylindrical frame field only.

[*Solutions on page 32*]

7 Connection forms

Read *O'Neill: Chapter II, Section 7, pages 85–90.*

We now come to the main idea of Sections 5–8: expressing the derivatives of an arbitrary frame field in terms of the frame field itself. We want an analogue of the Frenet formulas:

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

We are looking, informally, for something that looks like

$$(\text{derivative of}) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = (\text{suitable matrix}) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}.$$

We know what sort of derivative we must use: the covariant derivative. Thus we aim to express the three covariant derivatives

$$\begin{pmatrix} \nabla_{\mathbf{v_p}} E_1 \\ \nabla_{\mathbf{v_p}} E_2 \\ \nabla_{\mathbf{v_p}} E_3 \end{pmatrix}$$

in terms of $E_1(\mathbf{p})$, $E_2(\mathbf{p})$, $E_3(\mathbf{p})$.

The preamble to Lemma 7.1 has these ideas written out in full. In matrix form the equations become

$$\begin{pmatrix} \nabla_{\mathbf{v_p}} E_1 \\ \nabla_{\mathbf{v_p}} E_2 \\ \nabla_{\mathbf{v_p}} E_3 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} E_1(\mathbf{p}) \\ E_2(\mathbf{p}) \\ E_3(\mathbf{p}) \end{pmatrix}.$$

In any particular case the coefficients c_{ij} can be calculated by the usual methods of orthonormal expansion. However, we shall develop a general method of calculating them from information about the frame field E_i, $i = 1, 2, 3$.

Lemma 7.1 Since the coefficients c_{ij} above are real numbers which depend on the tangent vector being used to differentiate the frame field, it is inevitable that the functions

$$\mathbf{v_p} \longmapsto c_{ij}(\mathbf{v_p})$$

will turn out to be 1-forms. Indeed, anticipation of this result was one of the main reasons for introducing 1-forms.

The proof of the lemma uses only the linearity properties of covariant differentiation.

The other main property of covariant differentiation, the Leibniz property, also has consequences for the coefficients. Just as with the Frenet formulas, the matrix of coefficients is skew-symmetric.

We can summarize the results of Lemma 7.1 and Theorem 7.2 in matrix form as

$$\nabla_V \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} (V) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

in which form the analogy with the Frenet formulas is fairly clear.

Note that, just as with the Frenet formulas, the proofs used orthonormal expansion, the frame properties and the Leibniz property of differentiation. Not only do the results look similar, so do the methods of obtaining them. The Leibniz property gives rise to the skew-symmetry and, hence, to the zeros down the main diagonal. The other zeros that appeared in the Frenet matrix were due to the very special way in which the Frenet frame T, N, B was chosen.

Calculating connection forms In any particular example of frame fields, the frame will often be given in terms of the natural frame field. *O'Neill* now shows how to calculate the connection forms directly, if you are given the coefficients expressing a frame field E_i, $i = 1, 2, 3$, in terms of the natural frame field.

The proof of Theorem 7.3 applies the definition of covariant derivative and some of the lemmas proved earlier. The method of proof is interesting but much less important than the result. Once again, the matrix form is probably the more memorable:

$$\omega = dA\,{}^tA.$$

The Frenet formulas Rather than leave you to deduce the Frenet formulas from the connection equations (page 91, Exercise 8 of *O'Neill*), we present here one way of doing the deduction.

Suppose that we have a unit-speed curve

$$\alpha : s \longmapsto \alpha(s),$$

and a frame field whose restriction to the curve gives the Frenet frame. That is, specifically, E_1, E_2 and E_3 is a frame field and

The frame field is defined on enough of $\mathbf{E}^3$ to include all of the route of α.

$$\begin{aligned} E_1(\alpha) &= T, \\ E_2(\alpha) &= N, \\ E_3(\alpha) &= B. \end{aligned}$$

Now we need the link between differentiation with respect to s and covariant derivatives. The link is provided by the result we proved earlier:

This is differentiation 'along the curve'.

$$\nabla_{\alpha'} V = (V(\alpha))'.$$

We have

$$\begin{aligned} T'(s) &= (E_1(\alpha(s)))' \\ &= \nabla_{\alpha'} E_1. \end{aligned}$$

But we know that

$$\alpha' = T,$$

so, doing similar calculations for N' and B', we have

$$\begin{aligned} T' &= \nabla_T T, \\ N' &= \nabla_T N, \\ B' &= \nabla_T B. \end{aligned}$$

Now we can compare the results given by the definitions of the Frenet apparatus with the expressions for the covariant derivatives from the connection equations.

From the connection equations,

$$\nabla_T T = \omega_{11}(T)\,T + \omega_{12}(T)\,N + \omega_{13}(T)\,B.$$

We know that $\omega_{11} = 0$ for any frame field and the definitions of κ and N imply that

$$T' = \kappa N.$$

Thus

$$\begin{aligned} \omega_{12}(T) &= \kappa, \\ \omega_{13}(T) &= 0. \end{aligned}$$

Note that the immediate consequences of the skew-symmetry of the connection equations are that

$$\begin{aligned} \omega_{31}(T) &= -\omega_{13}(T) = 0, \\ \omega_{21}(T) &= -\omega_{12}(T) = -\kappa. \end{aligned}$$

The remaining independent connection form is ω_{23}. We can find this by considering B' and the definition of τ as $-B' \cdot N$:

$$\begin{aligned}B' &= \nabla_T B \\ &= \omega_{31}(T)\,T + \omega_{32}(T)\,N + \omega_{33}(T)\,B \\ &= 0 - \tau N + 0.\end{aligned}$$

Thus

$$\omega_{31}(T) = 0, \quad \omega_{32}(T) = -\tau.$$

Hence

$$\omega_{23}(T) = -\omega_{32}(T) = -(-\tau) = \tau.$$

Reasonably enough, the connection forms are known only on the curve. Summarizing, we have

$$\omega_{12}(T) = \kappa, \; \omega_{13}(T) = 0, \; \omega_{23}(T) = \tau.$$

Note that there is no point in even asking questions about expressions such as $\nabla_N T$ etc. This is because moving in the N-direction from a point on the curve immediately takes you off the curve to where T is undefined.

This situation will appear again when we consider surfaces.

Exercise 7.1 *O'Neill*, page 90, Exercise 1.

Exercise 7.2 *O'Neill*, page 90, Exercise 2.

Exercise 7.3 *O'Neill*, page 90, Exercise 3.

Exercise 7.4 *O'Neill*, page 91, Exercise 4.

Exercise 7.5 *O'Neill*, page 91, Exercise 5. (*Hint*: Try applying the appropriate version of the Leibniz property to each of the terms $f_i E_i$.)

[*Solutions on page 33*]

8 The structural equations

Read *O'Neill: Chapter II, Section 8, pages 91–95.*

The work in this section is placed here for the sake of completeness. It will not actually be put to use until well into our study of surfaces. However, logically it belongs with the previous discussion of frame fields.

What we are going to do is to generalize the relationship between the natural frame field U_1, U_2, U_3 and the 1-forms dx, dy, dz.

We know, from *O'Neill*, Chapter I, that the 1-forms dx, dy and dz pick out the coordinates of a tangent vector, that is if

$$\mathbf{v_p} = v_1 U_1(\mathbf{p}) + v_2 U_2(\mathbf{p}) + v_3 U_3(\mathbf{p}),$$

then

$$dx(\mathbf{v_p}) = v_1, \quad dy(\mathbf{v_p}) = v_2, \quad dz(\mathbf{v_p}) = v_3.$$

Definition 8.1 This is, effectively, the generalization referred to above. Because the $E_i(\mathbf{p})$ form a frame at $\mathbf{p}$, we can use orthonormal expansion to give

$$\mathbf{v_p} = (\mathbf{v_p} \cdot E_1(\mathbf{p}))E_1(\mathbf{p}) + (\mathbf{v_p} \cdot E_2(\mathbf{p}))E_2(\mathbf{p}) + (\mathbf{v_p} \cdot E_3(\mathbf{p}))E_3(\mathbf{p}).$$

Thus

$$\theta_i(\mathbf{v_p}) = \mathbf{v_p} \cdot E_i(\mathbf{p})$$

is the ith coordinate of the vector part of $\mathbf{v_p}$. Thus θ_i does indeed pick out the E_i-component of $\mathbf{v_p}$.

Note that *O'Neill* slips quietly into applying the dual 1-forms to vector fields, rather than tangent vectors, without any comment. What he is doing is to apply the pointwise principle to define

$$\theta_i(V) : \mathbf{p} \longmapsto V(\mathbf{p}) \cdot E_i(\mathbf{p}).$$

This can be stated more succinctly as

$$\theta_i(V) = V \cdot E_i.$$

Suppose that we have a vector field on $\mathbf{E}^3$ defined in terms of the E_i by

$$V = f_1 E_1 + f_2 E_2 + f_3 E_3,$$

where the f_i, $i = 1, 2, 3$ are functions on $\mathbf{E}^3$. Then, from the above and the properties of a frame field, we have

$$\begin{aligned}\theta_i(V) &= V \cdot E_i\\ &= (f_1 E_1 + f_2 E_2 + f_3 E_3) \cdot E_i\\ &= f_i.\end{aligned}$$

Thus, just as θ_i picks out the E_i-component of a tangent vector, θ_i picks out the E_i-coordinate function of a vector field.

A consequence of the definition of the dual 1-forms is that

$$\theta_i(E_j) = E_j \cdot E_i = \delta_{ij}.$$

This consequence could, in fact, be taken as the *definition* of the dual 1-forms.

Once we know the effect of a 1-form on a frame field, it is uniquely determined, by linearity (see Lemma 8.2 below). Thus, the dual 1-forms are completely determined by the relationships

$$\theta_i(E_j) = E_j \cdot E_i = \delta_{ij}.$$

This relationship, between a frame field and the dual 1-forms, is the easiest to use if you are asked to check that given 1-forms actually are the duals of a given frame field.

Lemma 8.2 This is the generalization of the result from *O'Neill*, Chapter I, that any 1-form can be expressed uniquely as a linear combination of dx, dy and dz. The lemma also provides a way of calculating the coefficients: evaluate the given 1-form on the frame field members.

Next, *O'Neill* shows how the dual 1-forms for a frame field are related to the 'standard' 1-forms dx, dy and dz. The relationship turns out to be exactly the same as between the frame field and the 'standard' frame field, that is, the natural frame field. The *same* attitude matrix does for both.

If the large number of summation signs confuses you, try writing out some of the expressions in full. For example, in the proof that

$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = A \begin{pmatrix}dx\\ dy\\ dz\end{pmatrix},$$

where A is the attitude matrix for the frame, a crucial step is the following.

$$\begin{aligned}\theta_i(U_j) &= U_j \cdot E_i\\ &= U_j \cdot (a_{i1}U_1 + a_{i2}U_2 + a_{i3}U_3)\\ &= a_{ij}.\end{aligned}$$

We have used the fact that $U_i \cdot U_j = \delta_{ij}$. The only term in the sum that survives is the one containing $U_j \cdot U_j$.

The structural equations The equations discovered by Cartan really are quite remarkable.

The discussion above, and in *O'Neill*, shows that the attitude matrix A describes all the 'static' information about a frame field: the frame field itself and its associated dual 1-forms. What the Cartan equations say is that the connection form matrix, ω, describes the rates of change of *everything* associated with a frame field: the covariant derivatives of the frame field, the exterior derivatives of the dual 1-forms and even the derivative of ω itself!

The attitude matrix gives the new frame and new 1-forms in terms of the 'standard' ones.

We think that it is instructive to consider proofs of the various parts of Theorem 8.3 that make more use of matrix methods. First we consider part (1).

In matrix form we have

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = A \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

and, by applying A^{-1} to both sides,

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = A^{-1} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}.$$

Because A is orthogonal, $A^{-1} = {}^tA$ so

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = {}^tA \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}.$$

Differentiating the original expression for the dual 1-forms,

$$\begin{aligned} \begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} &= dA \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \\ &= dA\,{}^tA \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \\ &= \omega \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}. \end{aligned}$$

With care we can give an even more concise matrix proof of part (2). We start from

$$\omega = dA\,{}^tA.$$

Applying the Leibniz property for exterior derivatives (*O'Neill*, Chapter I) we get

$$\begin{aligned} d\omega &= d(dA\,{}^tA) \\ &= d(dA)\,{}^tA - dA\,d({}^tA) \\ &= 0 - dA\,{}^t(dA). \end{aligned}$$

It is tedious, but not difficult, to show that the property also applies to matrices. You just have to consider the elements of the matrices.

To complete the proof the trick is to insert the identity matrix in the form

$$I = A^{-1}\,A = {}^tA\,A.$$

This gives

$$\begin{aligned} d\omega &= -(dA)\,({}^tA\,A)\,{}^tdA \\ &= -(dA\,{}^tA)(A\,{}^tdA) \\ &= -\omega\,{}^t(dA\,{}^tA) \\ &= -\omega\,{}^t\omega \\ &= \omega\,\omega. \quad \text{(Skew-symmetry of } \omega\text{, i.e. } {}^t\omega = -\omega.) \end{aligned}$$

Note that ${}^t(AB) = {}^tB\,{}^tA$ for any appropriately sized matrices A and B.

The family resemblance between the results that we have obtained in this section is all the more striking if they are all expressed in matrix form.

Where necessary, multiplication should be interpreted as using wedge products.

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = A \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix},$$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = A \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix},$$

$$\nabla_V \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \omega(V) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix},$$

$$d \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \omega \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix},$$

$$d\omega = \omega\,\omega.$$

Because we shall not be using these results for some time, we have not set many exercises on this section.

Since some of the exercises require you to show that two 1-forms are equal, we think that a reminder of how to do so is in order.

Being functions, two 1-forms are equal if, and only if, they have the same domain and the same values on all tangent vectors in their domains. Because 1-forms are linear, all that we need to check is that they agree on a basis at each point. In practice this means showing that they agree on a frame field.

To see why this is sufficient, suppose that ϕ and ψ are 1-forms, that E_i, $i = 1, 2, 3$ is a frame field and

$$\phi(E_i) = \psi(E_i), \quad i = 1, 2, 3.$$

If V is any vector field, then

$$V = v_1 E_1 + v_2 E_2 + v_3 E_3.$$

Then

$$\begin{aligned} \phi(V) &= \phi(v_1 E_1 + v_2 E_2 + v_3 E_3) \\ &= v_1 \phi(E_1) + v_2 \phi(E_2) + v_3 \phi(E_3) \\ &= v_1 \psi(E_1) + v_2 \psi(E_2) + v_3 \psi(E_3) \\ &= \psi(V). \end{aligned}$$

Thus ϕ and ψ have the same value on any vector field and are, therefore, equal as 1-forms.

Exercise 8.1 *O'Neill*, page 95, Exercise 1.

Exercise 8.2 *O'Neill*, page 96, Exercise 4.

Suggestion: The method advocated in *O'Neill* leads to rather long-winded calculations. We suggest that you start from the equation connecting the dual 1-forms and dx, dy and dz. Using $x = r\cos\vartheta$ and $y = r\sin\vartheta$ you can express the standard 1-forms in terms of dr, etc.

[*Solutions on page 34*]

9 Summary

Read *O'Neill: Chapter II, Section 9, page 96.*

The main purpose of this part has been to introduce you to the study of curves and the derivation of the Frenet formulas. However, the method of moving frames is of such general importance that we have also included the generalization of the Frenet approach to obtain the connection forms and Cartan's structural equations.

The most important computational techniques that you should be able to carry out are the calculation of the Frenet apparatus for unit-speed and arbitrary-speed curves.

The connection and structural equations will reappear in our study of surfaces.

For a unit-speed curve, the Frenet apparatus gives a description of the curve using only ideas belonging to the curve itself, rather than the description of the curve by three coordinate functions. The parameter s, the arc-length, belongs to the curve, not to where the curve is situated in $\mathbf{E}^3$. Similarly, the curvature κ and torsion τ are functions of s and, in principle at least, a creature confined to the curve, and having no knowledge of how the curve is situated in $\mathbf{E}^3$, could measure s, κ and τ. Such a creature would have to have perception of more than the one dimension of the curve in order to conceive the vectors N and B. Even so, this knowledge need be 'local' only to its current position on the curve; it need have no perception of the curve as a whole.

Nevertheless, s and the two functions κ and τ do give information about what the curve looks like to an observer who can see the curve embedded in $\mathbf{E}^3$.

In Part III (*O'Neill*, Chapter III) we show that knowledge of κ and τ as functions of s is enough to determine everything about a curve *except* its precise location in $\mathbf{E}^3$.

Solutions to the exercises

Solution 1.1

(a) Applying the definition:

$$\mathbf{p}\cdot\mathbf{q} = 1\times(-1)+2\times 0+(-1)\times 3 = -4.$$

(b) We assume that the tangent vectors are at a point $\mathbf{p}$. Using the formal determinant method:

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} U_1(\mathbf{p}) & U_2(\mathbf{p}) & U_3(\mathbf{p}) \\ 1 & 2 & -1 \\ -1 & 0 & 3 \end{vmatrix}$$

$$= U_1(\mathbf{p})\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} - U_2(\mathbf{p})\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} + U_3(\mathbf{p})\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix}$$

$$= (6,-2,2)_{\mathbf{p}}.$$

(c) Since $\|\mathbf{v}\| = \sqrt{6}$ and $\|\mathbf{w}\| = \sqrt{10}$, we have

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{6}}(1,2,-1),$$

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{1}{\sqrt{10}}(-1,0,3).$$

(d) Using the result of part (b),

$$\|\mathbf{v}\times\mathbf{w}\| = \sqrt{36+4+4} = \sqrt{44} = 2\sqrt{11}.$$

(e) If the angle is θ, we have

$$\begin{aligned}\cos\theta &= \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} \\ &= \frac{-4}{\sqrt{6}\times\sqrt{10}} \\ &= -\frac{4}{\sqrt{2}\sqrt{3}\sqrt{2}\sqrt{5}} \\ &= -\frac{2}{\sqrt{15}}.\end{aligned}$$

Solution 1.2

We must show that the basis is orthonormal, that is

$$e_i\cdot e_j = \delta_{ij}.$$

By direct calculation

$$\begin{aligned}
e_1\cdot e_1 &= \frac{1}{6}(1+4+1) = 1,\\
e_1\cdot e_2 &= \frac{1}{\sqrt{48}}(-2+0+2) = 0,\\
e_1\cdot e_3 &= \frac{1}{\sqrt{18}}(1-2+1) = 0,\\
e_2\cdot e_2 &= \frac{1}{8}(4+0+4) = 1,\\
e_2\cdot e_3 &= \frac{1}{\sqrt{24}}(-2+0+2) = 0,\\
e_3\cdot e_3 &= \frac{1}{3}(1+1+1) = 1.
\end{aligned}$$

Orthonormal expansion requires the coefficients:

$$\begin{aligned}
\mathbf{v}\cdot e_1 &= \frac{1}{\sqrt{6}}(6+2-1) = \frac{7}{\sqrt{6}},\\
\mathbf{v}\cdot e_2 &= \frac{1}{\sqrt{8}}(-12+0-2) = \frac{-14}{\sqrt{8}} = \frac{-7}{\sqrt{2}},\\
\mathbf{v}\cdot e_3 &= \frac{1}{\sqrt{3}}(6-1-1) = \frac{4}{\sqrt{3}}.
\end{aligned}$$

Thus, with a check:

$$\begin{aligned}
\mathbf{v} &= \frac{7}{\sqrt{6}}e_1 - \frac{14}{\sqrt{8}}e_2 + \frac{4}{\sqrt{3}}e_3\\
&= \tfrac{1}{6}(7,14,7) - \tfrac{1}{8}(-28,0,28) + \tfrac{1}{3}(4,-4,4)\\
&= \tfrac{1}{24}(28+84+32, 56+0-32, 28-84+32)\\
&= \tfrac{1}{24}(144,24,-24)\\
&= (6,1,-1).
\end{aligned}$$

Solution 1.3

(a) From the definition of cross product:

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} U_1(\mathbf{p}) & U_2(\mathbf{p}) & U_3(\mathbf{p}) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= U_1(\mathbf{p})\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - U_2(\mathbf{p})\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + U_3(\mathbf{p})\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}.$$

Hence

$$\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$$

$$= u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

(b) We apply the properties of the dot product and the rule about interchanging rows changing the sign.

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} &= \mathbf{v}\times\mathbf{w}\cdot\mathbf{u}\\
&= -\mathbf{v}\times\mathbf{u}\cdot\mathbf{w}\\
&= -(-\mathbf{u}\times\mathbf{v}\cdot\mathbf{w})\\
&= \mathbf{u}\times\mathbf{v}\cdot\mathbf{w}.
\end{aligned}$$

Solution 1.4

For the first part, if $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent, then the angle between them is 0 or π. Hence the sine of the angle is 0 and the length of the cross product is 0. Provided that neither vector is zero, each step in the argument is reversible and the result follows. If either is zero, then the vectors are automatically linearly dependent.

The area property follows from

$$\|\mathbf{v}\times\mathbf{w}\| = \|\mathbf{v}\|\,\|\mathbf{w}\|\sin\theta,$$

which is the area of the specified parallelogram.

Solution 1.5

Since we are dealing with a frame, $e_2\times e_3$ is perpendicular to e_2 and to e_3 and so is a multiple of e_1. Also

$$\begin{aligned}\|e_2\times e_3\| &= \|e_2\|\,\|e_3\|\sin(\pi/2)\\ &= 1.\end{aligned}$$

Thus,

$$e_2\times e_3 = \pm e_1$$

and

$$e_1\cdot e_2\times e_3 = \pm e_1\cdot e_1 = \pm 1.$$

Since the definition of orthogonal matrix is equivalent to saying that the rows (or columns) form a frame, the above shows that the determinant of such a matrix is ± 1.

Solution 1.6

Let

$$\mathbf{v}_1 = (\mathbf{v}\cdot\mathbf{u})\,\mathbf{u} \text{ and } \mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1.$$

Then

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \text{ and } \mathbf{v}_1\cdot\mathbf{v} = \mathbf{v}_1\cdot\mathbf{v}_1 + \mathbf{v}_1\cdot\mathbf{v}_2.$$

Now, substituting for $\mathbf{v}_1$ in the first two occurrences above,

$$\begin{aligned}
&(\mathbf{v}\cdot\mathbf{u})\,\mathbf{u}\cdot\mathbf{v} = ((\mathbf{v}\cdot\mathbf{u})\,\mathbf{u})\cdot((\mathbf{v}\cdot\mathbf{u})\,\mathbf{u}) + \mathbf{v}_1\cdot\mathbf{v}_2\\
&\Rightarrow (\mathbf{v}\cdot\mathbf{u})^2 = (\mathbf{v}\cdot\mathbf{u})^2\mathbf{u}\cdot\mathbf{u} + \mathbf{v}_1\cdot\mathbf{v}_2\\
&\Rightarrow (\mathbf{v}\cdot\mathbf{u})^2 = (\mathbf{v}\cdot\mathbf{u})^2 + \mathbf{v}_1\cdot\mathbf{v}_2 \quad (\mathbf{u} \text{ is a unit vector})\\
&\Rightarrow 0 = \mathbf{v}_1\cdot\mathbf{v}_2.
\end{aligned}$$

Thus $\mathbf{v}$ has an expression of the given form.

Conversely, $\mathbf{v}_1$ is uniquely defined by

$$\mathbf{v}_1 = (\mathbf{v}\cdot\mathbf{u})\,\mathbf{u},$$

hence so is

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1.$$

Solution 2.1

(a) Finding the derivatives of the coordinate functions of α involves repeated use of the Leibniz property. First, the velocity.

$$\alpha'(t) = (\cos t - t\sin t, \sin t + t\cos t, 1)_{\alpha(t)}.$$

Note that the velocity has been given as a tangent vector based at $\alpha(t)$. Next, the speed, this is the norm of $\alpha'(t)$.

$$\begin{aligned}
\|\alpha'(t)\|^2 &= (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1\\
&= \cos^2 t + t^2\sin^2 t - 2t\cos t\sin t\\
&\quad + \sin^2 t + t^2\cos^2 t + 2t\sin t\cos t + 1\\
&= 2 + t^2;
\end{aligned}$$

so

$$\|\alpha'(t)\| = \sqrt{2+t^2}.$$

Finally, the acceleration.

$$\alpha''(t) = (-2\sin t - t\cos t, 2\cos t - t\sin t, 0)_{\alpha(t)}.$$

(b) Since $\alpha(0) = (0,0,0)$, the route of the curve passes through the origin. At $t=0$, we have the velocity given by

$$\alpha'(0) = (1,0,1)_{(0,0,0)}$$

and so the speed is $\sqrt{2}$. Finally, the acceleration is

$$\alpha''(0) = (0,2,0)_{(0,0,0)}.$$

Solution 2.2

(a) We apply the technique outlined in the comments on the proof of Theorem 2.1.

We first find the velocity and hence the speed. Here, as later, we shall omit the point of application of the velocity.

$$\begin{aligned}
\alpha'(t) &= (\sinh t, \cosh t, 1).\\
\|\alpha'(t)\| &= \sqrt{\sinh^2 t + \cosh^2 t + 1}\\
&= \sqrt{2\cosh^2 t} \quad (\text{using } 1+\sinh^2 t = \cosh^2 t)\\
&= \sqrt{2}\cosh t.
\end{aligned}$$

Now

$$\begin{aligned}
s(t) &= \int_0^t \sqrt{2}\cosh u\, du\\
&= \sqrt{2}\,[\sinh u]_0^t\\
&= \sqrt{2}\sinh t.
\end{aligned}$$

Next, we have to find t in terms of s. The method discussed in Part 0 yields

$$t = \log_e\left(\frac{s+\sqrt{s^2+2}}{\sqrt{2}}\right).$$

However, in finding $\alpha(t(s))$, we also need $\sinh t$ and $\cosh t$ and we can obtain these directly from

$$s = \sqrt{2}\sinh t.$$

We have

$$\sinh t = \frac{s}{\sqrt{2}}$$

and

$$\begin{aligned}
\cosh^2 t &= 1 + \sinh^2 t\\
&= 1 + \frac{s^2}{2}\\
&= \frac{2+s^2}{2}.
\end{aligned}$$

Hence

$$\cosh t = \frac{\sqrt{2+s^2}}{\sqrt{2}}.$$

Finally

$$\begin{aligned}
\beta(s) &= \alpha(t(s))\\
&= \left(\frac{\sqrt{s^2+2}}{\sqrt{2}}, \frac{s}{\sqrt{2}}, \log_e\left(\frac{s+\sqrt{s^2+2}}{\sqrt{2}}\right)\right).
\end{aligned}$$

Solution 2.3

To show that the route of the curve passes through the specified points, we look at the first coordinate. If the route is to pass through $(2,1,0)$, it must be for $t=1$. Checking:

$$\alpha(1) = (2,1,0).$$

Similarly,

$$\alpha(2) = (4,4,\log_e 2).$$

The velocity is given by

$$\alpha'(t) = \left(2, 2t, \frac{1}{t}\right), \quad t>0,$$

and the speed by

$$\begin{aligned}
\|\alpha'(t)\| &= \sqrt{4+4t^2+\frac{1}{t^2}}\\
&= \sqrt{\left(2t+\frac{1}{t}\right)^2}\\
&= 2t + \frac{1}{t}.
\end{aligned}$$

Hence, the arc-length from $t=1$ to $t=2$ is

$$\begin{aligned}
\int_1^2 \left(2u+\frac{1}{u}\right) du &= \left[u^2 + \log_e u\right]_1^2\\
&= 4 + \log_e 2 - 1 - 0\\
&= 3 + \log_e 2.
\end{aligned}$$

Solution 2.4

The easiest approach to this problem is probably to use coordinate functions. Suppose that

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$$

with similar notation for β.

The given information amounts to saying that

$$\alpha_i'(t) = \beta_i'(t), \quad i = 1, 2, 3.$$

Now, all the coordinate functions are simply ordinary, real functions. Two such functions have equal derived functions if, and only if, they differ by a constant. Thus, there must be constants c_1, c_2 and c_3 such that

$$\beta_i(t) = \alpha_i(t) + c_i, \quad i = 1, 2, 3.$$

If we define $\mathbf{p}$ to be the point

$$\mathbf{p} = (c_1, c_2, c_3),$$

then we have shown that

$$\beta(t) = \alpha(t) + \mathbf{p}.$$

This means that the route of β is the route of α translated by the vector $\mathbf{p}$.

Solution 3.1

The easiest approach is to follow the layout of the definitions of the Frenet apparatus. First, though, we check that β is indeed unit speed.

$$\beta'(s) = (-\tfrac{4}{5}\sin s, -\cos s, \tfrac{3}{5}\sin s)$$

$$\begin{aligned}\|\beta'(s)\|^2 &= \tfrac{16}{25}\sin^2 s + \cos^2 s + \tfrac{9}{25}\sin^2 s\\ &= \sin^2 s + \cos^2 s\\ &= 1.\end{aligned}$$

Thus β is unit speed.

Now we apply the definitions of the Frenet apparatus.

$$\begin{aligned}T &= \beta'(s)\\ &= (-\tfrac{4}{5}\sin s, -\cos s, \tfrac{3}{5}\sin s);\\ T' &= (-\tfrac{4}{5}\cos s, \sin s, \tfrac{3}{5}\cos s),\\ \kappa &= \|T'\|\\ &= \sqrt{\tfrac{16}{25}\cos^2 s + \sin^2 s + \tfrac{9}{25}\cos^2 s}\\ &= \sqrt{\cos^2 s + \sin^2 s} = 1;\\ N &= \frac{T'}{\kappa}\\ &= (-\tfrac{4}{5}\cos s, \sin s, \tfrac{3}{5}\cos s);\\ B &= T \times N\\ &= (-\tfrac{3}{5}, 0, -\tfrac{4}{5});\\ B' &= 0;\\ \tau &= -B' \cdot N = 0.\end{aligned}$$

The results above show that β is a plane curve ($\tau = 0$), and has constant curvature 1. Thus (the route of) β is a circle of radius 1.

There are a number of methods of finding the centre. Perhaps the most elementary is to suppose that the centre is $\mathbf{p} = (p_1, p_2, p_3)$ and write down the equations that result from expressing the fact that three selected points on the curve are all 1 unit from $\mathbf{p}$. If we take the points

$$\beta(0) = (\tfrac{4}{5}, 1, -\tfrac{3}{5}),$$

$$\beta(\pi/2) = (0, 0, 0)$$

and

$$\beta(\pi) = (-\tfrac{4}{5}, 1, \tfrac{3}{5})$$

then this approach yields the three equations

$$\begin{aligned}(p_1 - 4/5)^2 + (p_2 - 1)^2 + (p_3 + 3/5)^2 &= 1,\\ p_1^2 + p_2^2 + p_3^2 &= 1,\\ (p_1 + 4/5)^2 + (p_2 - 1)^2 + (p_3 - 3/5)^2 &= 1.\end{aligned}$$

Expanding the first and third equations and subtracting the second from both results yields

$$\begin{aligned}-(8/5)p_1 - 2p_2 + (6/5)p_3 + 2 &= 0,\\ (8/5)p_1 - 2p_2 - (6/5)p_3 + 2 &= 0.\end{aligned}$$

Adding, we have

$$-4p_2 + 4 = 0,$$

and so $p_2 = 1$.

Substituting in the second of the original equations gives

$$p_1^2 + p_3^2 = 0.$$

Since squares are non-negative, this forces $p_1 = p_3 = 0$ and so the centre is $\mathbf{p} = (0, 1, 0)$.

Alternatively, we can use the remark on page 62 of *O'Neill*, just after the proof of Corollary 3.5. There it says that the unit normal N points towards the centre. Thus, we can start at any point and move one unit along the normal. If we start at

$$\beta(0) = (\tfrac{4}{5}, 0, -\tfrac{3}{5})$$

and travel one unit along

$$N(0) = (-\tfrac{4}{5}, 0, \tfrac{3}{5}),$$

we arrive at

$$\beta(0) + N(0) = (0, 1, 0)$$

as before.

We can confirm that this centre is correct by considering

$$\beta(s) - (0, 1, 0) = (\tfrac{4}{5}\cos s, -\sin s, -\tfrac{3}{5}\cos s).$$

We have

$$\begin{aligned}\|\beta(s) - (0, 1, 0)\|^2 &= \frac{16}{25}\cos^2 s + \sin^2 s + \frac{9}{25}\cos^2 s\\ &= \cos^2 s + \sin^2 s = 1.\end{aligned}$$

Thus $\beta(s)$ is a constant distance 1 from the point $(0, 1, 0)$. Thus the route is a circle, radius 1, centre $(0, 1, 0)$.

Solution 3.2

We proceed as in the first part of the last solution.

$$\begin{aligned}\beta'(s) &= \left(\frac{3}{2}\frac{(1+s)^{1/2}}{3}, -\frac{3}{2}\frac{(1-s)^{1/2}}{3}, \frac{1}{\sqrt{2}}\right)\\ &= \left(\frac{(1+s)^{1/2}}{2}, -\frac{(1-s)^{1/2}}{2}, \frac{1}{\sqrt{2}}\right);\end{aligned}$$

$$\|\beta'(s)\|^2 = \tfrac{1}{4}(1+s) + \tfrac{1}{4}(1-s) + \tfrac{1}{2} = 1.$$

Hence, β is unit speed. It follows that

$$\begin{aligned}T &= \beta'(s)\\ &= \left(\frac{(1+s)^{1/2}}{2}, -\frac{(1-s)^{1/2}}{2}, \frac{1}{\sqrt{2}}\right);\\ T' &= \left(\tfrac{1}{4}(1+s)^{-1/2}, \tfrac{1}{4}(1-s)^{-1/2}, 0\right),\end{aligned}$$

$$
\begin{aligned}
\kappa &= \|T'\| \\
&= \tfrac{1}{4}\sqrt{(1+s)^{-1}+(1-s)^{-1}} \\
&= \tfrac{1}{4}\sqrt{\frac{1-s+1+s}{(1+s)(1-s)}} \\
&= \tfrac{1}{4}\sqrt{\frac{2}{1-s^2}} \\
&= \frac{1}{2\sqrt{2}\sqrt{1-s^2}};
\end{aligned}
$$

$$
\begin{aligned}
N &= \frac{T'}{\kappa} \\
&= 4\sqrt{\frac{1-s^2}{2}}\left(\tfrac{1}{4}(1+s)^{-1/2}, \tfrac{1}{4}(1-s)^{-1/2}, 0\right); \\
&= \frac{1}{\sqrt{2}}\left((1-s)^{1/2}, (1+s)^{1/2}, 0\right);
\end{aligned}
$$

$$
\begin{aligned}
B &= T \times N \\
&= \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}(1+s)^{1/2}, \frac{1}{\sqrt{2}}(1-s)^{1/2}, \frac{1+s}{2}+\frac{1-s}{2}\right) \\
&= \tfrac{1}{2}\left(-(1+s)^{1/2}, (1-s)^{1/2}, \sqrt{2}\right);
\end{aligned}
$$

$$
\begin{aligned}
B' &= \tfrac{1}{2}\left(-\tfrac{1}{2}(1+s)^{-1/2}, -\tfrac{1}{2}(1-s)^{-1/2}, 0\right) \\
&= \tfrac{1}{4}\left(-(1+s)^{-1/2}, -(1-s)^{-1/2}, 0\right);
\end{aligned}
$$

$$
\begin{aligned}
\tau &= -B'\cdot N \\
&= -\frac{1}{4\sqrt{2}}\left(-\sqrt{\frac{1-s}{1+s}}-\sqrt{\frac{1+s}{1-s}}\right) \\
&= \frac{1}{4\sqrt{2}}\left(\sqrt{\frac{1-s}{1+s}}+\sqrt{\frac{1+s}{1-s}}\right) \\
&= \frac{1}{4\sqrt{2}}\left(\frac{(1-s)+(1+s)}{\sqrt{(1+s)(1-s)}}\right) \\
&= \frac{1}{4\sqrt{2}}\frac{2}{\sqrt{1-s^2}} \\
&= \frac{1}{2\sqrt{2}\sqrt{1-s^2}}.
\end{aligned}
$$

Note that, since $\tau=\kappa$, the ratio

$$\frac{\tau}{\kappa}=1$$

is constant. As will be shown in the next section, this is sufficient to show that β is a cylindrical helix.

Solution 3.3

We simply compute the three cross products and show that they give the right-hand sides of the respective Frenet formulas and, hence, the three derivatives.

$$
\begin{aligned}
A\times T &= (\tau T+\kappa B)\times T \\
&= \tau 0+\kappa N \\
&= T';
\end{aligned}
$$

$$
\begin{aligned}
A\times N &= (\tau T+\kappa B)\times N \\
&= \tau B+\kappa(-T) \\
&= N';
\end{aligned}
$$

$$
\begin{aligned}
A\times B &= (\tau T+\kappa B)\times B \\
&= \tau(-N)+\kappa 0 \\
&= B'.
\end{aligned}
$$

Solution 3.4

We write down the three requirements

$$\gamma(0)=\beta(0),\quad \gamma'(0)=\beta'(0),\quad \gamma''(0)=\beta''(0)$$

and see what can be deduced by use of the Frenet formulas.

First, we calculate the values of γ and its first two derivatives at 0:

$$
\begin{aligned}
\gamma(0) &= \mathbf{c}+r\mathbf{e}_1, \\
\gamma'(s) &= -\sin(s/r)\mathbf{e}_1+\cos(s/r)\mathbf{e}_2, \\
\gamma'(0) &= \mathbf{e}_2 \\
\gamma''(s) &= -(1/r)\cos(s/r)\mathbf{e}_1-(1/r)\cos(s/r)\mathbf{e}_2, \\
\gamma''(0) &= -(1/r)\mathbf{e}_1.
\end{aligned}
$$

Next, we do the same for the derivatives of β.

$$
\begin{aligned}
\beta'(s) &= T(s), \\
\beta'(0) &= T(0), \\
\beta''(s) &= T'(s) \\
&= \kappa(s)N(s), \\
\beta''(0) &= \kappa(0)N(0).
\end{aligned}
$$

Now we equate the various values that we have obtained.

$$
\begin{aligned}
\mathbf{c}+r\mathbf{e}_1 &= \beta(0), \\
\mathbf{e}_2 &= T(0), \\
-(1/r)\mathbf{e}_1 &= \kappa(0)N(0).
\end{aligned}
$$

We are aiming to find r, $\mathbf{e}_1$ and $\mathbf{e}_2$ in terms of the Frenet apparatus of β. Taking norms in the third equation above gives

$$|(1/r)|=|\kappa(0)|,\quad (\|N\|=\|\mathbf{e}_1\|=1).$$

Since r and κ are positive, we have

$$r=1/\kappa(0).$$

The third equation also shows that $\mathbf{e}_1$ and $N(0)$ are unit vectors in opposite directions, that is,

$$\mathbf{e}_1=-N(0).$$

The second equation shows directly that

$$\mathbf{e}_2=T(0).$$

Finally,

$$
\begin{aligned}
\mathbf{c} &= \beta(0)-r\mathbf{e}_1 \\
&= \beta(0)-(1/\kappa(0))(-N(0)) \\
&= \beta(0)+(1/\kappa(0))N(0).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\gamma(s) = \beta(0) &+ (1/\kappa(0))(1-\cos(s\kappa(0)))N(0) \\
&+ (1/\kappa(0))\sin(s\kappa(0))T(0).
\end{aligned}
$$

This shows that γ lies in the T–N plane at $\beta(0)$, that is, the osculating plane at $\beta(0)$.

Solution 3.5

(a) There are two approaches suggested by what we have done. The first uses the same techniques as were used for the Frenet formulas. We express N' using orthonormal expansion:

$$N'=(N'\cdot T)T+(N'\cdot N)N.$$

Now we use the fact that N and T are orthogonal unit vectors, together with the Leibniz property.

$$
\begin{aligned}
& N\cdot N=1 \\
\Rightarrow\ & 2N'\cdot N=0 \\
\Rightarrow\ & N'\cdot N=0.
\end{aligned}
$$

$$T \cdot N = 0$$
$$\Rightarrow T' \cdot N + T \cdot N' = 0$$
$$\Rightarrow N' \cdot T = -T' \cdot N.$$

Since

$$-T' \cdot N = -(\kappa N) \cdot N = -\kappa,$$

substituting for $N' \cdot T$ and $N' \cdot N$ in the orthonormal expansion gives

$$N' = -\kappa T$$

as required.

We could also tackle the problem directly. Using the notation in the question, and the fact that $T' = \kappa N$, we have

$$\begin{aligned} T' &= (x'', y'') \\ &= \kappa(-y', x') \\ &= (-\kappa y', \kappa x'). \end{aligned}$$

Thus $x'' = -\kappa y'$ and $y'' = \kappa x'$.

Hence

$$\begin{aligned} N' &= (-y'', x'') \\ &= (-\kappa x', -\kappa y') \\ &= -\kappa(x', y') \\ &= -\kappa T. \end{aligned}$$

(b) Note that, as *O'Neill* says in the footnote, ϕ is the angle between T and the x-axis.

Probably the easiest way to tackle this is to express T' and N in terms of ϕ. Because of the definition of N, we have

$$N = -\sin\phi\, U_1 + \cos\phi\, U_2.$$

Differentiating the expression given for T with respect to s gives

$$T' = -\sin\phi\, \phi'(s)\, U_1 + \cos\phi\, \phi'(s)\, U_2.$$

(We have used the chain rule here.)

Comparing T' and κN, gives

$$\kappa = \phi'.$$

Intuitively, this result is reasonable. Curvature is the rate of turning of the tangent, which is precisely what ϕ' measures.

Solution 4.1

We carry out the calculations as indicated by Theorem 4.3. We calculate the derivatives of α, the cross and triple products and then use the formulas from the theorem.

$$\begin{aligned} \alpha'(t) &= (2, 2t, t^2); \\ \alpha''(t) &= (0, 2, 2t); \\ \alpha'''(t) &= (0, 0, 2). \\ \alpha'(t) \times \alpha''(t) &= (2t^2, -4t, 4), \\ \|\alpha'(t) \times \alpha''(t)\| &= \sqrt{4t^4 + 16t^2 + 16} \\ &= \sqrt{4(t^2+2)^2} \\ &= 2(2 + t^2). \end{aligned}$$

$$\alpha'(t) \times \alpha''(t) \cdot \alpha'''(t) = 8.$$

Hence

$$\begin{aligned} v &= \|\alpha'(t)\| \\ &= \sqrt{4 + 4t^2 + t^4} \\ &= \sqrt{(2+t^2)^2} \\ &= 2 + t^2. \end{aligned}$$

It follows that

$$\begin{aligned} T &= \frac{1}{v}\alpha'(t) \\ &= \frac{1}{2+t^2}(2, 2t, t^2). \end{aligned}$$

Next

$$\kappa v^3 = \|\alpha'(t) \times \alpha''(t)\|.$$

Hence

$$\begin{aligned} \kappa &= \frac{2(2+t^2)}{v^3} \\ &= \frac{2(2+t^2)}{(2+t^2)^3} \\ &= \frac{2}{(2+t^2)^2}. \end{aligned}$$

Now B:

$$\begin{aligned} B &= \frac{\alpha'(t) \times \alpha''(t)}{\|\alpha'(t) \times \alpha''(t)\|} \\ &= \frac{1}{2(2+t^2)}(2t^2, -4t, 4) \\ &= \frac{1}{2+t^2}(t^2, -2t, 2). \end{aligned}$$

$$\begin{aligned} N &= B \times T \\ &= \frac{1}{(2+t^2)^2}\left(-2t(t^2+2), 4 - t^4, 2t(t^2+2)\right) \\ &= \frac{1}{2+t^2}(-2t, 2 - t^2, 2t). \end{aligned}$$

Finally,

$$\begin{aligned} \tau &= \frac{\alpha'(t) \times \alpha''(t) \cdot \alpha'''(t)}{\|\alpha'(t) \times \alpha''(t)\|^2} \\ &= \frac{8}{4(2+t^2)^2} = \frac{2}{(2+t^2)^2}. \end{aligned}$$

Note that $\tau/\kappa = 1$, so that the curve is a cylindrical helix.

Solution 4.2

We approach this in the same way as the last question.

$$\begin{aligned} \alpha'(t) &= (\sinh t, \cosh t, 1), \\ \alpha''(t) &= (\cosh t, \sinh t, 0), \\ \alpha'''(t) &= (\sinh t, \cosh t, 0). \end{aligned}$$

$$\begin{aligned} \alpha'(t) \times \alpha''(t) &= (-\sinh t, \cosh t, \sinh^2 t - \cosh^2 t) \\ &= (-\sinh t, \cosh t, -1), \\ &\quad (\text{because } \cosh^2 t - \sinh^2 t = 1); \end{aligned}$$

$$\begin{aligned} \|\alpha'(t) \times \alpha''(t)\| &= \sqrt{\sinh^2 t + \cosh^2 t + 1} \\ &= \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t. \end{aligned}$$

$$\begin{aligned} \alpha'(t) \times \alpha''(t) \cdot \alpha'''(t) &= -\sinh^2 t + \cosh^2 t \\ &= 1. \end{aligned}$$

$$\begin{aligned} v &= \|\alpha'(t)\| \\ &= \sqrt{\sinh^2 t + \cosh^2 t + 1} \\ &= \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t. \end{aligned}$$

$$
\begin{aligned}
T &= \frac{1}{v}\alpha'(t) \\
&= \frac{1}{\sqrt{2}\cosh t}(\sinh t, \cosh t, 1). \\
\kappa &= \frac{\|\alpha'(t) \times \alpha''(t)\|}{v^3} \\
&= \frac{\sqrt{2}\cosh t}{2\sqrt{2}\cosh^3 t} = \frac{1}{2\cosh^2 t}. \\
B &= \frac{\alpha'(t) \times \alpha''(t)}{\|\alpha'(t) \times \alpha''(t)\|} \\
&= \frac{1}{\sqrt{2}\cosh t}(-\sinh t, \cosh t, -1). \\
N &= B \times T \\
&= \frac{1}{2\cosh^2 t}(2\cosh t, 0, 2\sinh t \cosh t) \\
&= \frac{1}{\cosh t}(1, 0, \sinh t). \\
\tau &= \frac{\alpha'(t) \times \alpha''(t) \cdot \alpha'''(t)}{\|\alpha'(t) \times \alpha''(t)\|^2} \\
&= \frac{1}{2\cosh^2 t}.
\end{aligned}
$$

Once again, this is a cylindrical helix.

To complete the question, we must find the arc-length function. We have

$$
\begin{aligned}
s(t) &= \int_0^t v(u)\,du \\
&= \int_0^t \sqrt{2}\cosh u\,du \\
&= \left[\sqrt{2}\sinh u\right]_0^t = \sqrt{2}\sinh t.
\end{aligned}
$$

Since

$$\cosh^2 t = 1 + \sinh^2 t = 1 + (s/\sqrt{2})^2 = 1 + s^2/2,$$

we have

$$
\begin{aligned}
\kappa &= \tau \\
&= \frac{1}{2(1 + s^2/2)} \\
&= \frac{1}{2 + s^2}.
\end{aligned}
$$

Solution 4.3

Following the suggestion in the question, all derivatives are evaluated at $t = 0$ before calculating cross products etc.

$$
\begin{aligned}
\alpha'(t) &= (\cos t - t\sin t, \sin t + t\cos t, 1), \\
\alpha''(t) &= (-2\sin t - t\cos t, 2\cos t - t\sin t, 0), \\
\alpha'''(t) &= (-3\cos t + t\sin t, -3\sin t - t\cos t, 0).
\end{aligned}
$$

$$
\begin{aligned}
\alpha'(0) &= (1, 0, 1), \\
\alpha''(0) &= (0, 2, 0), \\
\alpha'''(t) &= (-3, 0, 0). \\
\alpha'(0) \times \alpha''(0) &= (-2, 0, 2) \\
\|\alpha'(0) \times \alpha''(0)\| &= \sqrt{4 + 0 + 4} \\
&= 2\sqrt{2}.
\end{aligned}
$$

$$\alpha'(0) \times \alpha''(0) \cdot \alpha'''(0) = 6.$$

$$
\begin{aligned}
v(0) &= \sqrt{1 + 0 + 1} \\
&= \sqrt{2}.
\end{aligned}
$$

$$
\begin{aligned}
T(0) &= \frac{\alpha'(0)}{v(0)} \\
&= \frac{1}{\sqrt{2}}(1, 0, 1). \\
\kappa(0) &= \frac{\|\alpha'(0) \times \alpha''(0)\|}{v(0)^3} = \frac{2\sqrt{2}}{2\sqrt{2}} = 1. \\
B(0) &= \frac{\alpha'(0) \times \alpha''(0)}{\|\alpha'(0) \times \alpha''(0)\|} \\
&= \frac{1}{2\sqrt{2}}(-2, 0, 2) \\
&= \frac{1}{\sqrt{2}}(-1, 0, 1). \\
N &= B \times T \\
&= \tfrac{1}{2}(0, 2, 0) = (0, 1, 0). \\
\tau(0) &= \frac{\alpha'(0) \times \alpha''(0) \cdot \alpha'''(0)}{\|\alpha'(0) \times \alpha''(0)\|^2} \\
&= \tfrac{6}{8} = \tfrac{3}{4}.
\end{aligned}
$$

Solution 4.4

The solution below follows the hint given.

$$
\begin{aligned}
\alpha'(t) &= cT, \\
\alpha''(t) &= cT' \quad (c \text{ constant}) \\
&= c(c\kappa N) \quad \text{(Frenet)} \\
&= c^2\kappa N. \\
\alpha'''(t) &= c^2\kappa' N + c^2\kappa N' \\
&\qquad \text{(we may not assume that } \kappa \text{ is constant)} \\
&= c^2\kappa' N + c^2\kappa(-c\kappa T + c\tau B) \quad \text{(Frenet)} \\
&= -c^3\kappa^2 T + c^2\kappa' N + c^3\kappa\tau B.
\end{aligned}
$$

The first line immediately gives the first result:

$$T = \frac{\alpha'}{c}.$$

The expression for α'' shows that it and N are in the same direction and so

$$N = \frac{\alpha''}{\|\alpha''\|}.$$

It follows that

$$
\begin{aligned}
B &= T \times N \\
&= \frac{\alpha'}{c} \times \frac{\alpha''}{\|\alpha''\|} \\
&= \frac{\alpha' \times \alpha''}{c\|\alpha''\|}.
\end{aligned}
$$

By taking the norm of α'' we obtain

$$\|\alpha''\| = c^2\kappa$$

and, hence,

$$\kappa = \frac{\|\alpha''\|}{c^2}.$$

Now we calculate the triple product.

$$
\begin{aligned}
\alpha' \times \alpha'' &= cT \times c^2\kappa N \\
&= c^3\kappa T \times N \\
&= c^3\kappa B. \\
\alpha' \times \alpha'' \cdot \alpha''' &= 0 + 0 + c^6\kappa^2\tau.
\end{aligned}
$$

Since

$$
\begin{aligned}
c^6\kappa^2 &= c^6\frac{\|\alpha''\|^2}{c^4} \\
&= c^2\|\alpha''\|^2,
\end{aligned}
$$

we have

$$\begin{aligned}\tau &= \frac{\alpha' \times \alpha'' \cdot \alpha'''}{c^6 \kappa^2} \\ &= \frac{\alpha' \times \alpha'' \cdot \alpha'''}{c^2 \|\alpha''\|^2}.\end{aligned}$$

Solution 4.5

First of all we show that the route of γ passes through $\alpha(0)$ and then that the tangent to γ is always perpendicular to $\mathbf{u}$.

$$\begin{aligned}\gamma(0) &= \alpha(0) - s(0)\cos\vartheta\,\mathbf{u} \\ &= \alpha(0)\end{aligned}$$

since $s(0) = 0$ from the information given.

Now

$$\begin{aligned}\gamma(t) &= \alpha(t) - s(t)\cos\vartheta\,\mathbf{u}, \\ \gamma'(t) &= \alpha'(t) - s'(t)\cos\vartheta\,\mathbf{u} \\ &= vT - v\cos\vartheta\,\mathbf{u}. \\ \gamma'(t)\cdot\mathbf{u} &= vT\cdot\mathbf{u} - v\cos\vartheta\,\mathbf{u}\cdot\mathbf{u} \\ &= v\cos\vartheta - v\cos\vartheta \\ &= 0.\end{aligned}$$

In the above we have used the fact that $\mathbf{u}$ is a constant unit vector, that ϑ is constant and that $T\cdot\mathbf{u} = \cos\vartheta$.

We have now shown that γ passes through $\alpha(0)$ and its tangent is always orthogonal to $\mathbf{u}$. That is sufficient to prove the assertion in part (a).

We now calculate the curvature of γ; as suggested we deal only with the case that α is unit speed. We also make use of the expression for $\mathbf{u}$:

$$\mathbf{u} = \cos\vartheta\, T + \sin\vartheta\, B.$$

From the above we have

$$\begin{aligned}\gamma'(t) &= vT - v\cos\vartheta\,\mathbf{u} \\ &= T - \cos\vartheta\,\mathbf{u}, \quad \text{(unit speed)}. \\ \|\gamma'(t)\|^2 &= (T - \cos\vartheta\,\mathbf{u})\cdot(T - \cos\vartheta\,\mathbf{u}) \\ &= T\cdot T - 2\cos\vartheta\,\mathbf{u}\cdot T + \cos^2\vartheta\,\mathbf{u}\cdot\mathbf{u} \\ &= 1 - \cos^2\vartheta \\ &= \sin^2\vartheta. \\ \|\gamma'(t)\| &= \sin\vartheta. \\ \gamma''(t) &= T' - 0 \\ &= \kappa N. \\ \gamma'(t)\times\gamma''(t) &= (T - \cos\vartheta\,\mathbf{u})\times\kappa N \\ &= \kappa(T\times N - \cos\vartheta\,\mathbf{u}\times N) \\ &= \kappa\,(B - \cos\vartheta(\cos\vartheta\,T + \sin\vartheta\,B)\times N) \\ &= \kappa\,\big(B(1-\cos^2\vartheta) + \cos\vartheta\sin\vartheta\,T\big) \\ &= \kappa\sin\vartheta(\cos\vartheta\,T + \sin\vartheta\,B) \\ &= \kappa\sin\vartheta\,\mathbf{u}. \\ \|\gamma'(t)\times\gamma''(t)\| &= \kappa\sin\vartheta. \\ \kappa_\gamma &= \frac{\kappa\sin\vartheta}{\sin^3\vartheta} = \frac{\kappa}{\sin^2\vartheta}.\end{aligned}$$

Solution 4.6

We apply the results of the previous exercise.

First, because τ/κ is clearly a constant, the curve is a cylindrical helix.

The cross-section curve has curvature

$$\frac{\kappa}{\sin^2\vartheta}$$

which is, from the given information, a constant. The cross-section curve is, therefore, a plane curve of constant curvature. That is, (part of) a circle. The cylinder on which the curve lies has a circular cross-section and the curve is, therefore, a circular helix.

Solution 4.7

Making use of the expression derived in *O'Neill* for σ'', we have

$$\begin{aligned}\sigma''' &= -2\kappa\kappa' T - \kappa^2 T' + \kappa'' N + \kappa' N' + (\kappa\tau)' B + \kappa\tau B' \\ &= -2\kappa\kappa' T - \kappa^3 N + \kappa'' N + \kappa'(-\kappa T + \tau B) \\ &\quad + (\kappa'\tau + \kappa\tau')B - \kappa\tau^2 N \\ &= -3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B.\end{aligned}$$

$$\begin{aligned}\sigma'\times\sigma''\cdot\sigma''' &= (\kappa^2\tau)(-3\kappa\kappa') + \kappa^3(2\kappa'\tau + \kappa\tau') \\ &= \kappa^3(\kappa\tau' - \kappa'\tau).\end{aligned}$$

Since

$$\left(\frac{\tau}{\kappa}\right)' = \frac{\kappa\tau' - \kappa'\tau}{\kappa^2},$$

we have

$$\begin{aligned}\sigma'\times\sigma''\cdot\sigma''' &= \kappa^3\kappa^2\left(\frac{\tau}{\kappa}\right)' \\ &= \kappa^5\left(\frac{\tau}{\kappa}\right)'.\end{aligned}$$

We also have

$$\begin{aligned}\|\sigma'\times\sigma''\| &= \kappa^2\sqrt{\kappa^2+\tau^2} \\ &= \kappa^3\sqrt{1+(\tau/\kappa)^2}.\end{aligned}$$

Thus

$$\begin{aligned}\tau_\sigma &= \frac{\sigma'\times\sigma''\cdot\sigma'''}{\|\sigma'\times\sigma''\|^2} \\ &= \frac{\kappa^5(\tau/\kappa)'}{\kappa^6\,(1+(\tau/\kappa)^2)} \\ &= \frac{(\tau/\kappa)'}{\kappa\,(1+(\tau/\kappa)^2)}.\end{aligned}$$

Solution 4.8

We can apply all the results obtained so far about spherical images.

A unit-speed curve β is a cylindrical helix if, and only if, the ratio τ_β/κ_β is constant. Suppose that this constant is c and the spherical image is σ. Then, if β is a cylindrical helix,

$$\kappa_\sigma = \sqrt{1+c^2}$$

which is constant. Further

$$\tau_\sigma = \frac{(d/ds)(c)}{\kappa_\beta(1+c^2)} = 0.$$

Thus σ is a plane ($\tau = 0$) curve of constant curvature, that is part of a circle.

Conversely, if the spherical image is part of a circle, then

$$\tau_\sigma = 0$$

and the expression for τ_σ shows that

$$\frac{d}{ds}\left(\frac{\tau_\beta}{\kappa_\beta}\right) = 0.$$

It follows that τ_β/κ_β is constant and β is a cylindrical helix.

Solution 5.1

Since the question said 'from first principles', we apply Definition 5.1 directly.

(a) We have

$$\begin{aligned}\nabla_{\mathbf{v_p}} W &= (W(1+t, 3-t, -1+2t))'(0)\\ &= \left((1+t)^2 U_1(\mathbf{p}+t\mathbf{v}) + (3-t)U_2(\mathbf{p}+t\mathbf{v})\right)'(0)\\ &= (2(1+t)U_1(\mathbf{p}+t\mathbf{v}) + (-1)U_2(\mathbf{p}+t\mathbf{v}))(0)\\ &= 2U_1(\mathbf{p}) - U_2(\mathbf{p}) \quad (= (2,-1,0)_{\mathbf{p}}).\end{aligned}$$

(b) In this case

$$\begin{aligned}\nabla_{\mathbf{v_p}} W &= \big((1+t)U_1(\mathbf{p}+t\mathbf{v}) + (1+t)^2 U_2(\mathbf{p}+t\mathbf{v})\\ &\qquad -(-1+2t)^2 U_3(\mathbf{p}+t\mathbf{v})\big)'(0)\\ &= (U_1(\mathbf{p}+t\mathbf{v}) + 2(1+t)U_2(\mathbf{p}+t\mathbf{v})\\ &\qquad -4(-1+2t)U_3(\mathbf{p}+t\mathbf{v}))(0)\\ &= U_1(\mathbf{p}) + 2U_2(\mathbf{p}) + 4U_3(\mathbf{p}) \quad (= (1,2,4)_{\mathbf{p}}).\end{aligned}$$

Solution 5.2

The approach here is to use the linearity properties of the covariant derivative, together with the basic result

$$\nabla_{U_i} f U_j = \frac{\partial f}{\partial x_i} U_j.$$

(a) This is a straightforward application of both varieties of linearity from Corollary 5.4.

$$\begin{aligned}\nabla_V W &= \nabla_{-yU_1 + xU_3} W\\ &= -y\nabla_{U_1} W + x\nabla_{U_3} W \quad \text{(linearity)}\\ &= -y\,(\nabla_{U_1} \cos x\, U_1 + \nabla_{U_1} \sin x\, U_2)\\ &\qquad + x\,(\nabla_{U_3} \cos x\, U_1 + \nabla_{U_3} \sin x\, U_2) \quad \text{(linearity)}\\ &= -y\,(-\sin x\, U_1 + \cos x\, U_2) + x\,(0+0)\\ &= y(\sin x\, U_1 - \cos x\, U_2).\end{aligned}$$

(b) Here we first apply Corollary 5.4(3) and then use the above result.

$$\begin{aligned}\nabla_V z^2 W &= V[z^2]W + z^2\nabla_V W\\ &= (-yU_1 + xU_3)[z^2]\,W + z^2 y(\sin xU_1 - \cos xU_2)\\ &= (-y\times 0 + x\times 2z)(\cos xU_1 + \sin xU_2)\\ &\qquad + z^2 y(\sin xU_1 - \cos xU_2)\\ &= (2xz\cos x + yz^2\sin x)U_1\\ &\qquad + (2xz\sin x - yz^2\cos x)U_2.\end{aligned}$$

(c) We make use of the result from the first part of this question.

$$\begin{aligned}\nabla_V\nabla_V W &= \nabla_V y(\sin x\, U_1 - \cos x\, U_2)\\ &= -y\nabla_{U_1} y(\sin x\, U_1 - \cos x\, U_2)\\ &\qquad + x\nabla_{U_3} y(\sin x\, U_1 - \cos x\, U_2)\\ &= -y\,(y(\cos x\, U_1 + \sin x\, U_2) + x(0+0)\\ &= -y^2(\cos x\, U_1 + \sin x\, U_2) \quad (= -y^2 W).\end{aligned}$$

Solution 5.3

Suppose that V has coordinate functions v_i for $i = 1, 2, 3$. That is,

$$V = \sum_{i=1}^{3} v_i U_i.$$

Then, by linearity,

$$\nabla_V X = \sum_{i=1}^{3} v_i \nabla_{U_i} X.$$

But

$$\begin{aligned}\nabla_{U_i} X &= \nabla_{U_i}(x_1U_1 + x_2U_2 + x_3U_3)\\ &= U_i\end{aligned}$$

because

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij}.$$

Hence

$$\nabla_V X = \sum_{i=1}^{3} v_i U_i = V.$$

Solution 5.4

We begin by using the definition to calculate ∇W.

$$\begin{aligned}\nabla W &= d(xy^3)U_1 - d(x^2z^2)U_3\\ &= (y^3\,dx + 3xy^2\,dy)U_1 - (2xz^2\,dx + 2x^2z\,dz)U_3.\end{aligned}$$

Now, using the definition given and the methods of Chapter I, we can do the evaluations.

(a) We have

$$\begin{aligned}\nabla_{\mathbf{v_p}} W &= (\nabla W)(\mathbf{v_p})\\ &= \left((y^3\,dx + 3xy^2\,dy)U_1 - (2xz^2\,dx + 2x^2z\,dz)U_3\right)(\mathbf{v_p})\\ &= (8(1) + 3(-1)(4)(0))U_1(\mathbf{p})\\ &\qquad - (2(-1)(1)(1) + 2(1)(-1)(-3))U_3(\mathbf{p})\\ &= 8U_1(\mathbf{p}) - 4U_3(\mathbf{p}).\end{aligned}$$

(b) This time, the same method leads to

$$\nabla_{\mathbf{v_p}} W = 27U_1(\mathbf{p}) + 12U_3(\mathbf{p}).$$

Solution 5.5

We have

$$\begin{aligned}\alpha(0) &= (1+\sin(0), 3-\sin(0), -1)\\ &= (1, 3, -1).\\ \alpha'(t) &= (\cos t, -\cos t, 2t+2).\\ \alpha'(0) &= (1, -1, 2).\end{aligned}$$

Next

$$W(\alpha(t)) = (1+\sin t)^2 U_1(\alpha(t)) + (3-\sin t)U_2(\alpha(t)).$$

So

$$\begin{aligned}\nabla_{(1,-1,2)_{(1,3,-1)}} W &= (W(\alpha(t)))'(0)\\ &= (2\cos t(1+\sin t)U_1(\alpha(t))\\ &\qquad - \cos tU_2(\alpha(t)))(0)\\ &= 2U_1(1,3,-1) - U_2(1,3,-1).\end{aligned}$$

This is exactly the same result as we obtained in the first solution of this set. Since, using the notation from that solution,

$$\begin{aligned}\alpha(0) &= \mathbf{p}\\ \alpha'(0) &= \mathbf{v}\end{aligned}$$

this is to be expected.

Solution 6.1

Since we are given the alleged frame, all we have to do is check that

$$E_i \cdot E_j = \delta_{ij}.$$

$$E_1 \cdot E_1 = \frac{V\cdot V}{\|V\|^2} = \frac{\|V\|^2}{\|V\|^2} = 1.$$

A similar calculation shows that $E_2 \cdot E_2 = 1$.

Now we show that the vector fields E_1 and E_2 are orthogonal.

$$\begin{aligned} E_1 \cdot E_2 &= \frac{E_1 \cdot (W - (W \cdot E_1)E_1)}{\|\tilde{W}\|} \\ &= \frac{(W \cdot E_1) - (W \cdot E_1)(E_1 \cdot E_1)}{\|\tilde{W}\|} \\ &= \frac{(W \cdot E_1) - (W \cdot E_1) \times 1}{\|\tilde{W}\|} \\ &= 0. \end{aligned}$$

Since E_1 and E_2 are orthogonal unit vector fields, their cross product E_3 is orthogonal to both of them and

$$\|E_3\| = 1 \times 1 \times \sin(\pi/2) = 1.$$

This completes the proof that the given fields form a frame field.

Solution 6.2

We apply Lemma 6.3, that is orthonormal expansion. We repeatedly use the fact that

$$U_i \cdot U_j = \delta_{ij}.$$

(a) Here

$$\begin{aligned} U_1 \cdot E_1 &= U_1 \cdot (\cos\vartheta\, U_1 + \sin\vartheta\, U_2) \\ &= \cos\vartheta, \\ U_1 \cdot E_2 &= U_1 \cdot (-\sin\vartheta\, U_1 + \cos\vartheta\, U_2) \\ &= -\sin\vartheta, \\ U_1 \cdot E_3 &= U_1 \cdot U_3 \\ &= 0. \\ U_1 &= \cos\vartheta\, E_1 - \sin\vartheta\, E_2. \end{aligned}$$

(b) We can save quite a bit of effort by noting that the first two terms are just E_1, so the given vector field is

$$E_1 + E_3.$$

Direct calculation of dot products gives exactly the same result, using $\sin^2\vartheta + \cos^2\vartheta = 1$.

(c) Orthonormal expansion gives

$$\begin{aligned} (x\,U_1 + y\,U_2 + z\,U_3) \cdot E_1 &= x\cos\vartheta + y\sin\vartheta \\ &= r\cos^2\vartheta + r\sin^2\vartheta \\ &= r, \\ (x\,U_1 + y\,U_2 + z\,U_3) \cdot E_2 &= -x\sin\vartheta + y\cos\vartheta \\ &= r\cos\vartheta\sin\vartheta + r\sin\vartheta\cos\vartheta \\ &= 0, \\ (x\,U_1 + y\,U_2 + z\,U_3) \cdot E_3 &= z. \end{aligned}$$

Hence

$$x\,U_1 + y\,U_2 + z\,U_3) = rE_1 + zE_3.$$

Solution 7.1

The easiest way of tackling such a question is probably to work with the attitude matrix A, say. We can check that we have a frame by showing that

$$A^tA = I,$$

and then use Theorem 7.3 to calculate the connection forms.

Here

$$A = \begin{pmatrix} \sin f/\sqrt{2} & 1/\sqrt{2} & -\cos f/\sqrt{2} \\ \sin f/\sqrt{2} & -1/\sqrt{2} & -\cos f/\sqrt{2} \\ \cos f & 0 & \sin f \end{pmatrix},$$

so

$$\begin{aligned} A^tA &= \begin{pmatrix} \sin f/\sqrt{2} & 1/\sqrt{2} & -\cos f/\sqrt{2} \\ \sin f/\sqrt{2} & -1/\sqrt{2} & -\cos f/\sqrt{2} \\ \cos f & 0 & \sin f \end{pmatrix} \\ &\quad \times \begin{pmatrix} \sin f/\sqrt{2} & \sin f/\sqrt{2} & \cos f \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -\cos f/\sqrt{2} & -\cos f/\sqrt{2} & \sin f \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

after repeated use of $\sin^2 f + \cos^2 f = 1$.

In order to apply Theorem 7.3, we need to calculate the entries in dA. We use the techniques from *O'Neill*, Chapter I, for example, the chain rule gives

$$d(\sin f) = \cos f\, df.$$

Thus

$$\begin{aligned} \omega &= dA^tA \\ &= \begin{pmatrix} (\cos f/\sqrt{2})\, df & 0 & (\sin f/\sqrt{2})\, df \\ (\cos f/\sqrt{2})\, df & 0 & (\sin f/\sqrt{2})\, df \\ -\sin f\, df & 0 & \cos f\, df \end{pmatrix} \\ &\quad \times \begin{pmatrix} \sin f/\sqrt{2} & \sin f/\sqrt{2} & \cos f \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -\cos f/\sqrt{2} & -\cos f/\sqrt{2} & \sin f \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & (1/\sqrt{2})\, df \\ 0 & 0 & (1/\sqrt{2})\, df \\ -(1/\sqrt{2})\, df & -(1/\sqrt{2})\, df & 0 \end{pmatrix}. \end{aligned}$$

Solution 7.2

Since the attitude matrix A of the natural frame field contains only constants, the derivative dA is the zero matrix. Thus

$$\omega = dA^tA = 0.$$

This result is not surprising since each vector field making up the natural frame field is parallel. As you move in any direction the natural frame field does not rotate at all so the connection forms must be identically zero.

Solution 7.3

A direct calculation shows that $A^tA = I$. However, you may have noticed a similarity with the spherical frame field. A careful study of the spherical frame field on page 83 of *O'Neill* shows that the following transforms the spherical case to the one in hand:

- interchange F_2 and F_3,
- substitute $\vartheta = f$,
- substitute $\phi = f$,
- change the sign of the new F_2.

None of these operations affect the property of being a frame.

The calculation of ω is long but not too difficult. The main step is the calculation of dA. Careful differentiation yields

$$dA = \begin{pmatrix} -2\sin f\cos f\, df & (\cos^2 f - \sin^2 f)\, df & \cos f\, df \\ (\cos^2 f - \sin^2 f)\, df & 2\sin f\cos f\, df & \sin f\, df \\ -\cos f\, df & -\sin f\, df & 0 \end{pmatrix}$$

Using this gives

$$\omega = \begin{pmatrix} 0 & -df & \cos f\, df \\ df & 0 & \sin f\, df \\ -\cos f\, df & -\sin f\, df & 0 \end{pmatrix}.$$

Solution 7.4

The attitude matrix A is

$$\begin{pmatrix} \cos\phi\cos\vartheta & \cos\phi\sin\vartheta & \sin\phi \\ -\sin\vartheta & \cos\vartheta & 0 \\ -\sin\phi\cos\vartheta & -\sin\phi\sin\vartheta & \cos\phi \end{pmatrix}.$$

Differentiating yields the following for dA.

$$\begin{pmatrix} -\sin\phi\cos\vartheta\, d\phi & -\sin\phi\sin\vartheta\, d\phi & \cos\phi\, d\phi \\ -\cos\phi\sin\vartheta\, d\vartheta & +\cos\phi\cos\vartheta\, d\vartheta & \\ & & \\ -\cos\vartheta\, d\vartheta & -\sin\vartheta\, d\vartheta & 0 \\ & & \\ -\cos\phi\cos\vartheta\, d\phi & -\cos\phi\sin\vartheta\, d\phi & -\sin\phi\, d\phi \\ +\sin\phi\sin\vartheta\, d\vartheta & -\sin\phi\cos\vartheta\, d\vartheta & \end{pmatrix}.$$

Use of $\omega = dA^tA$ and repeated use of $\sin^2 + \cos^2 = 1$ gives

$$\omega = \begin{pmatrix} 0 & \cos\phi\, d\vartheta & d\phi \\ -\cos\phi\, d\vartheta & 0 & \sin\phi\, d\vartheta \\ -d\phi & -\sin\phi\, d\vartheta & 0 \end{pmatrix}.$$

Solution 7.5

Linearity enables us to consider each of the terms f_iE_i separately, since

$$\nabla_V \sum_{j=1}^{3} f_jE_j = \sum_{j=1}^{3} \nabla_V f_jE_j.$$

Now

$$\begin{aligned} \nabla_V f_iE_i &= V[f_i]E_i + f_i\nabla_V E_i \quad \text{(Leibniz)} \\ &= V[f_i]E_i + f_i \sum_{j=1}^{3} \omega_{ij}(V)E_j, \\ &\qquad \text{(connection equations)}. \end{aligned}$$

To obtain the result in the form given in the question involves summing the result above for $i = 1, 2, 3$ and manipulating the double sum.

$$\begin{aligned} \nabla_V W &= \sum_{i=1}^{3} \left(V[f_i]E_i + f_i \sum_{j=1}^{3} \omega_{ij}(V)E_j \right) \\ &= \sum_{i=1}^{3} V[f_i]E_i + \sum_{i=1}^{3}\sum_{j=1}^{3} f_i\omega_{ij}(V)E_j. \end{aligned}$$

Relabelling the 'summation variable' in the first sum and interchanging the order of summation in the second term gives

$$\begin{aligned} \nabla_V W &= \sum_{j=1}^{3} V[f_j]E_j + \sum_{j=1}^{3}\sum_{i=1}^{3} f_i\omega_{ij}(V)E_j \\ &= \sum_{j=1}^{3} \left(V[f_j] + \sum_{i=1}^{3} f_i\omega_{ij}(V) \right) E_j. \end{aligned}$$

Note that this result generalizes the following.

If

$$W = \sum f_iU_i$$

then

$$\nabla_V W = \sum V[f_i]U_i.$$

In the case of the natural frame field the result is simpler precisely because the connection forms for the natural frame field are identically zero.

Solution 8.1

We follow the same plan as the corresponding exercise for vector fields in the last section. Linearity allows us to deal with each term separately.

$$\begin{aligned} d(f_i\theta_i) &= df_i \wedge \theta_i + f_i d\theta_i \\ &= df_i \wedge \theta_i + f_i \sum_{j=1}^{3} \omega_{ij} \wedge \theta_j. \end{aligned}$$

Summing:

$$\begin{aligned} d\phi &= \sum_{i=1}^{3} \left(df_i \wedge \theta_i + f_i \sum_{j=1}^{3} \omega_{ij} \wedge \theta_j \right) \\ &= \sum_{i=1}^{3} df_i \wedge \theta_i + \sum_{i=1}^{3}\sum_{j=1}^{3} f_i\omega_{ij} \wedge \theta_j \\ &= \sum_{j=1}^{3} df_j \wedge \theta_j + \sum_{j=1}^{3}\sum_{i=1}^{3} f_i\omega_{ij} \wedge \theta_j \\ &= \sum_{j=1}^{3} \left(df_i + \sum_{i=1}^{3} f_i\omega_{ij} \right) \wedge \theta_j. \end{aligned}$$

Solution 8.2

(a) We follow the hint in the text, rather than that in the question. We begin by using

$$x = r\cos\vartheta,$$
$$y = r\sin\vartheta,$$

to calculate dx and dy.

$$dx = d(r\cos\vartheta) = \cos\vartheta\, dr - r\sin\vartheta\, d\vartheta,$$
$$dy = d(r\sin\vartheta) = \sin\vartheta\, dr + r\cos\vartheta\, d\vartheta.$$

Let A be the attitude matrix of the cylindrical frame field, then

$$\begin{aligned} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} &= A \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \\ &= \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\vartheta\, dr - r\sin\vartheta\, d\vartheta \\ \sin\vartheta\, dr + r\cos\vartheta\, d\vartheta \\ dz \end{pmatrix} \\ &= \begin{pmatrix} (\cos^2\vartheta + \sin^2\vartheta)dr + 0d\vartheta \\ 0dr + r(\sin^2\vartheta + \cos^2\vartheta)d\vartheta \\ dz \end{pmatrix} \\ &= \begin{pmatrix} dr \\ rd\vartheta \\ dz \end{pmatrix}. \end{aligned}$$

From this, the results follow.

(b) We apply duality and the above results in the form

$$dr = \theta_1, \quad d\vartheta = \frac{1}{r}\theta_2, \quad dz = \theta_3.$$

Now,

$$E_1[r] = dr(E_1) = \theta_1(E_1) = 1,$$

by duality.

Similarly

$$\begin{aligned} E_2[\vartheta] &= d\vartheta(E_1) \\ &= \frac{1}{r}\theta_2(E_2) \\ &= \frac{1}{r}. \end{aligned}$$

Also

$$E_3[z] = dz(E_3) = \theta_3(E_3) = 1.$$

The remaining results are obtained in exactly the same way. As a sample, we do the following.

$$\begin{aligned} E_1[\vartheta] &= d\vartheta(E_1) \\ &= \frac{1}{r}\theta_2(E_1) = 0. \end{aligned}$$

(c) We can apply the results just obtained.

First, since $V[f] = df(V)$, for any vector field V, we calculate df in terms of the dual 1-forms.

$$\begin{aligned} df &= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \vartheta}d\vartheta + \frac{\partial f}{\partial z}dz \\ &= \frac{\partial f}{\partial r}\theta_1 + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\theta_2 + \frac{\partial f}{\partial z}\theta_3. \end{aligned}$$

Applying df to E_1 and using duality gives

$$\begin{aligned} df(E_1) &= \left(\frac{\partial f}{\partial r}\theta_1 + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\theta_2 + \frac{\partial f}{\partial z}\theta_3\right)(E_1) \\ &= \frac{\partial f}{\partial r} \times 1 + 0 + 0 \\ &= \frac{\partial f}{\partial r}. \end{aligned}$$

Similar calculations give the remaining results.

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